Solution 2 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let f be the function defined for $x > -1$ by $f(x) = \ln(1+x)$, and let a be a positive real number. Using the Mean Value Theorem, there is a real number $\xi \in (0, a)$ such that

$$\frac{\ln(1+a)}{a} = \frac{f(a) - f(0)}{a} = f'(\xi) = \frac{1}{1+\xi} \in \left(\frac{1}{1+a}, 1\right).$$

that is, for $a > 0$, we have

$$\frac{a}{1+a} < \ln(1+a) < a.$$

Applying the upper inequality with $a = 1/n$ and the lower one with $a = 1/m$, we get

$$n \ln\left(1 + \frac{1}{n}\right) < 1 < (m+1)\ln\left(1 + \frac{1}{m}\right)$$

Taking exponentials yields the desired inequality.

Also solved by Arkady Alt, San Jose, California, USA, Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain.

58. *Let $x = 2\cos A, y = 2\cos B$ and $z = 2\cos C$, where A, B, C are the measures of the angles of an acute triangle ABC. Find the minimum value of*

$$x^4 + y^4 + z^4 + x^2y^2z^2$$

(Training Catalonian Team for OME 2014)

Solution 1 by Arkady Alt, San Jose, California, USA. By replacing $(\cos A, \cos B, \cos C)$ in identity

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$$

with $\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$ we obtain

$$\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{4} + 2 \cdot \frac{x}{2} \cdot \frac{y}{2} \cdot \frac{z}{2} = 1$$

Thus

$$x^2 + y^2 + z^2 + xyz = 4$$

Since triangle ABC is acute, that is $x, y, z > 0$, then by QM-AM inequality we have $\frac{x^4 + y^4 + z^4 + x^2y^2z^2}{4} \geq \left(\frac{x^2 + y^2 + z^2 + xyz}{4}\right)^2 = 1 \implies x^4 + y^4 + z^4 + x^2y^2z^2 \geq 4$.

Since the lower bound 4 can be attained if $x = y = z \iff A = B = C$, the desired minimum is 4.

Solution 2 by José Luis Díaz-Barrero BARCELONA TECH, Barcelona, Spain. Since $A+B+C=\pi$, then we have

$$\begin{aligned} x^2+y^2+z^2+xyz &= 4\cos^2 A+4\cos^2 B+4\cos^2(A+B) \\ &\quad -8\cos A\cos B\cos(A+B) \\ &= 4(\cos^2 A+\cos^2 A-\cos^2 A\cos^2 A+\sin^2 A\sin^2 B) \\ &= 4\Big[\sin^2 B(\cos^2 A+\sin^2 A)+\cos^2 B\Big]=4 \end{aligned}$$

Taking into account AM-QM inequality yields

$$1=\frac{x^2+y^2+z^2+xyz}{4}\le\sqrt{\frac{x^4+y^4+z^4+x^2y^2z^2}{4}}$$

from which follows $x^4+y^4+z^4+x^2y^2z^2\ge 4$. So, the minimum value of the expression claimed is 4 and it is attained when $\triangle ABC$ is equilateral.

Solution 3 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. The answer is 4 and it is attained only when the triangle is equilateral.
First, note as in the preceding solutions we have

$$x^2+y^2+z^2+xyz=4$$

Thus,

$$\begin{aligned} x^4+y^4+z^4+x^2y^2z^2-4 &= x^4+y^4+z^4+x^2y^2z^2-2(x^2+y^2+z^2+xyz)+4 \\ &= (x^2-1)^2+(y^2-1)^2+(z^2-1)^2+(xyz-1)^2\ge 0 \end{aligned}$$

with equality if and only if $x^2=y^2=z^2=xyz=1$, or equivalently $A=B=C=60^\circ$.

Remark. Note that the condition that ABC is acute is unnecessary.

Solution 4 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy. It is a known standard result that $1\le\cos A+\cos B+\cos C\le 3/2$ and then $2\le 2\cos A+2\cos B+2\cos C\le 3$. The minimum 1 corresponds to a degenerate isosceles triangle while the maximum to an equilateral triangle. Clearly we have $2\le x+y+z\le 3$. We employ the so called "uvw" theory which can be found at The art of problem solving forum. Define three new quantities

$$x+y+z=3u,\quad xy+yz+zx=3v^2,\quad xyz=w^3$$

We have

$$x^4+y^4+z^4+x^2y^2z^2=(w^3)^2+12uw^3+81u^4-108u^2v^2+18v^4=(w^3)^2+12uw^3+R(u,v)$$

This is a convex increasing parabola if $w^3\ge 0$ whose minimum has negative abscissa. It follows that the minimum of the parabola occurs when $w=0$ or when w is minimum once fixed the values of u and v. According to the theory, the latter occurs when $x=y$ (or cyclic). If $w=0$ we have for instance $z=0$ that is $C=\pi/2$. At $x+y$ fixed, the minimum of x^4+y^4, occurs when $x=y$ that is $A=B=\pi/4$ or $z=y=\sqrt{2}$. This yields

$$x^4+y^4=2x^4=8$$