Value of polynomial generated by sequence (by the footprints of one class of problems of mathematical olympiads).

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The reason for these notes was a series of problems of the same type which at different times offered at mathematical olympiads and establishing a link (one of many) between numerical sequences and polynomials.)

All problems below presented in the original formulation, and in the order which is arbitrary except the first one on the list that reminded me of the existence of the rest.

Perhaps this list is not complete and there are a lot more of them, but this does not matter anymore since the main goal is to solve the problem, which is a generalization of all possible problems of this type with a demonstration all necessary for this technique.

Problem1.(Mathproblems, vol.4,n.1,2014, Mathcontest Section #64)

Let $A(x)$ be a polynomial with integer coefficients such that for $1 \leq k \leq n+1$ holds:

$$A(k) = 5^k$$

Find the value of $A(n+2)$.

Remark.

Formulation of the problem should include a requirement $\deg A(x) \leq n$ because otherwise the value of $A(n+2)$ can't be detrmined uniquely. Indeed, if for some polynomial $A(x)$ with integer coefficients holds $A(k) = 5^k,\ k = 1, 2, ..., n+1$ then for polynomial $P(x) = A(x) + (x-1)(x-2)...(x-n-1)Q(x)$,where $Q(x)$ is any polynomial we have

$P(k) = 5^k = A(k),\ k = 1, 2, ..., n+1$ as well, but value $P(n+2)$ will be vary with $Q(x)$.

Problem 2. (Short-Listed problems of IMO in Rumania 1983,#19)

Let $(F_n)_{n\geq 1}$ be the Fibonacci sequence $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$ $(n \geq 1)$,and $P(x)$ the polynomial of degree 990 satisfying $P(k) = F_k$, for $k = 992, ..., 1982$.

Prove that $P(1983) = F_{1983} - 1$.

Problem 3. (Short-Listed problems of IMO in USA 1981,#13).

Let $P(x)$ be polynimial of degree n such that

$$P(k) = 1/\binom{n+1}{k}, k = 0, 1, ..., n.\text{Find } P(n+1).$$

Problem 4. (USA 1975).

Let $P(x)$ be polynimial of degree n such that $P(k) = \dfrac{k}{k+1}, k = 0, 1, ..., n.$Find $P(n+1)$.

This list could be infinite, bearing in mind all possible sequences but we still confine ourselves to some finite subset of it, adding in the further from themself, a few more particular cases of the following basic problem.

Basic Problem.

For given sequence $a_0, a_1, ..., a_n, ...$ let $A_n(x), n = 0, 1, 2, ...$ be polynomial defined as follows:

$$\textbf{(1)} \qquad \begin{cases} A_n(k) = a_k, k = 0, 1, 2, ..., n \\ \deg A_n(x) \leq n \end{cases}.$$

Find the value of $A_n(n+1)$.

Remark.

In reality we are talking about sequence of polynomials $A_0(x), A_1(x), ..., A_n(x), ...$defined for each n by properties **(1).** Obvious that $A_0(x)$ is constant polynomial and $A_0(x) = a_0$. Also note that $A_n(k) = A_m(k) = a_k$ for any $k = 0, 1, 2, ..., m \leq n$.

Let's find $A_n(n+1)$ for $n = 1, 2, 3$.

Let $A_1(x) = \alpha_0 + \alpha_1 x$.Since $A_1(0) = a_0$ then $A_1(x) = a_0 + \alpha_1 x$ and, therefore, $A_1(1) = a_1 \iff a_0 + \alpha_1 = a_1 \iff \alpha_1 = a_1 - a_0$.

Thus, $A_1(x) = a_0 + (a_1 - a_0)x \implies A_1(2) = a_0 + 2(a_1 - a_0) = 2a_1 - a_0$.

We will find $A_2(x)$ represented in the form $A_2(x) = \alpha_0 + \alpha_1 x + \alpha_2 x(x-1)$ (because in the such form more convenient to find coefficients $\alpha_0, \alpha_1, \alpha_2$ than in usual presentation by powers of x). Since $A_2(x) = A_1(x) = \alpha_0 + \alpha_1 x$ for $x = 0, 1$ then $A_1(x) = \alpha_0 + \alpha_1 x$ for all x and, therefore, $A_2(x) = A_1(x) + \alpha_2 x(x-1)$.

Using $A_2(2) = a_2 \iff A_1(2) + \alpha_2 \cdot 2 \cdot (2-1) = a_2$ and $A_1(2) = 2a_1 - a_0$ we obtain

$$2!\alpha_2 + 2a_1 - a_0 = a_2 \iff \alpha_2 = \frac{a_2 - 2a_1 + a_0}{2!}.$$

Thus, $A_2(x) = A_1(x) + \dfrac{a_2 - 2a_1 + a_0}{2!} x(x-1) = a_0 + \dfrac{a_1 - a_0}{1!} x + \dfrac{a_2 - 2a_1 + a_0}{2!} x(x-1) \implies$

$$A_2(3) = a_0 + \frac{a_1 - a_0}{1!} \cdot 3 + \frac{a_2 - 2a_1 + a_0}{2!} \cdot 3(3-1) = a_0 - 3a_1 + 3a_2.$$

To go to the general case, we need to make some terminological preparation

For the sake of convenience and bearing in mind the notation for binomial coefficients we will, by analogy, denote the polynomial $\dfrac{x(x-1)(x-2)...(x-n+1)}{n!}$ via $\dbinom{x}{n}, n \in \mathbb{N}$ and by definition $\dbinom{x}{0} := 1$, $\dbinom{x}{-n} := 0, n \in \mathbb{N}$.

Note that $\dbinom{x}{n} = 0$ for all non-negative integers $x < n$.

Also, to emphasize that $A_n(n+1)$ isn't, generally speaking, the term of the sequence $(a_n)_{n \geq 0}$ we set $b_n := A_n(n+1)$, $n = 0, 1, 2, ...$

Thus, our main goal is to express b_n through the terms of the sequence $(a_n)_{n \geq 0}$ and, as a tribute to curiosity, find the polynomial $A_n(x)$.

For example, we already have $b_0 = a_0, b_1 = 2a_1 - a_0$, $b_2 = a_0 - 3a_1 + 3a_2$ and by the way, the polynomials $A_0(x), A_1(x), A_2(x)$.

Assume that we already have pollynomials $A_0(x), A_1(x), ..., A_n(x)$ which satisfy **(1)**.

We going to find $A_{n+1}(x)$ in the form $A_{n+1}(x) = P(x) + \alpha_{n+1} \dbinom{x}{n+1}$, where

$\deg P(x) \leq n$.

Since $A_{n+1}(x) = A_n(x) = P(x)$ for $x = 0, 1, 2, ..., n$ then $P(x) = A_n(x)$ for all x and, therefore, $A_{n+1}(x) = A_n(x) + \alpha_{n+1}\binom{x}{n+1}$, where coefficient α_{n+1} is determined by claim $A_{n+1}(n+1) = a_{n+1}$.

We have $a_{n+1} = A_{n+1}(n+1) = A_n(n+1) + \alpha_{n+1}\binom{n+1}{n+1} \iff$

$a_{n+1} = b_n + \alpha_{n+1} \iff \alpha_{n+1} = a_{n+1} - b_n$.

So, $A_{n+1}(x) = A_n(x) + (a_{n+1} - b_n)\binom{x}{n+1}$.

For any function $f(x)$ we define difference operator Δ as follows:

$\Delta f(x) = f(x+1) - f(x)$ and define recursively k-times iterated difference operator Δ^k :

$\Delta^0 f(x) = f(x)$ and $\Delta^k f(x) := \Delta\left(\Delta^{k-1} f(x)\right), k \in \mathbb{N}$.

Since $\Delta(c) = 0, \Delta\left(\binom{x}{1}\right) = \Delta(x) = 1$ and $\Delta\left(\binom{x}{n}\right) = \binom{x+1}{n} - \binom{x}{n} = \frac{(x+1)x(x-1)(x-2)...(x-n+2)}{n!} - \frac{x(x-1)(x-2)...(x-n+1)}{n!} = \frac{x(x-1)(x-2)...(x-n+2)}{n!}((x+1-(x-n+1))) = \frac{x(x-1)(x-2)...(x-n+2)n}{n!} = \binom{x}{n-1}$ then

$$\Delta^k\left(\binom{x}{n}\right) = \Delta^k\left(\binom{x}{n-k}\right) = \begin{cases} 0 \text{ if } k > n \\ 1 \text{ if } k = n \end{cases}.$$

Also note that Δ is linear operator, that is $\Delta(f(x) + g(x)) = \Delta f(x) + \Delta g(x)$ and $\Delta(cf(x)) = c\Delta f(x)$ .Then $\Delta^k, k \in \mathbb{N}$ is linear operator as well and, applying Δ^{n+1} to $A_{n+1}(x) = A_n(x) + \alpha_{n+1}\binom{x}{n+1}$, we obtain

$\Delta^{n+1}(A_{n+1}(x)) = \Delta^{n+1}(A_n(x)) + \Delta^{n+1}\left(\alpha_{n+1}\binom{x}{n+1}\right) \iff$

$\Delta^{n+1}(A_{n+1}(x)) = 0 + \alpha_{n+1} \implies \Delta^{n+1}(A_{n+1}(0)) = \alpha_{n+1} \iff$

$\Delta^{n+1}(a_0) = \alpha_{n+1}$ (since $\Delta^{n+1}(A_{n+1}(x))$ is constant polynomial).

Thus, $A_{n+1}(x) = A_n(x) + \Delta^{n+1}(a_0)\binom{x}{n+1}$ and, therefore,

$A_{n+1}(n+1) = A_n(n+1) + \Delta^{n+1}(a_0)\binom{n+1}{n+1} \iff a_{n+1} = b_n + \Delta^{n+1}(a_0) \iff b_n = a_{n+1} - \Delta^{n+1}(a_0)$.

Since $\Delta^{n+1}(a_0) = \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}a_{n+1-k}, n \in \mathbb{N}$ then

(2) $A_n(n+1) = b_n = a_{n+1} - \Delta^{n+1}(a_0) = a_{n+1} - \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}a_{n+1-k} = \sum_{k=1}^{n+1}(-1)^{k-1}\binom{n+1}{k}a_{n+1-k}$ and

(3) $A_n(x) = a_0 + \sum_{k=1}^{n}\Delta^k(a_0)\binom{x}{k} = \sum_{k=0}^{n}\Delta^k(a_0)\binom{x}{k}$.

Here it's time to digress a little and as an example solve problem **3**

using correlation $A_n\left(n+1\right)=\sum\limits_{k=1}^{n+1}\left(-1\right)^{k-1}\binom{n+1}{k}a_{n+1-k}$ for

$a_k=\dfrac{1}{\dbinom{n+1}{k}}=\dfrac{1}{\dbinom{n+1}{n+1-k}}=a_{n+1-k}$. $\left(P\left(n+1\right)=A_n\left(n+1\right)=\sum\limits_{k=1}^{n+1}\left(-1\right)^{k-1}\binom{n+1}{k}\dfrac{1}{\dbinom{n+1}{k}}=\right.$

$\left.\sum\limits_{k=1}^{n+1}\left(-1\right)^{k-1}=\dfrac{\left(-1\right)^n+1}{2}\right)$.

Thus, by the way, represented above, any sequence $a_{\mathbb{N}_*}:=\left(a_0,a_1,...,a_n,...\right)$ generate

sequence of polynomials $L\left(a_{\mathbb{N}_*}\right):=\left(A_0\left(x\right),A_1\left(x\right),...,A_n\left(x\right),...\right)$.

Easy to see that for any two sequences $a_{\mathbb{N}}$ and $b_{\mathbb{N}}$ holds $L\left(a_{\mathbb{N}}+b_{\mathbb{N}}\right)=L\left(a_{\mathbb{N}}\right)+L\left(b_{\mathbb{N}}\right)$

and for any constant c holds $L\left(ca_{\mathbb{N}}\right)=cL\left(a_{\mathbb{N}}\right)$ (linearity of operator L).

Linearity of L gives opportunity to find polynomial $A_n\left(x\right)$ and value of $A_n\left(n+1\right)$ for

sequence $\alpha a_n+\beta b_n$ by decomposition, that is by reducing to sequences a_n and b_n

that is $L\left(\alpha a_n+\beta b_n\right)=\alpha L\left(a_n\right)+\beta L\left(b_n\right)$.

Applications.(with adaptation for the concrete problem of notations used in the basic problem)

Problem 1 we will solve in more general form by replacing 5 in $A(k)=5^k$ with any $a\neq 1$.

Then $a_k=a^{k+1}$, $k=0,1,...,n$, $A(x)=A_n(x)$ and $A(n+2)=A_n(n+1)$.

Since $\Delta^k\left(a_0\right)=\sum\limits_{i=0}^{k}\left(-1\right)^i\binom{k}{i}a_{k-i}=\sum\limits_{i=0}^{k}\left(-1\right)^i\binom{k}{i}a^{k-i+1}=a\sum\limits_{i=0}^{k}\left(-1\right)^i\binom{k}{i}a^{k-i}=a\left(a-1\right)^k$

we have $A_n\left(x\right)=\sum\limits_{k=0}^{n}a\left(a-1\right)^k\binom{x}{k}=a\sum\limits_{k=0}^{n}\left(a-1\right)^k\binom{x}{k}$ and

$A(n+2)=A_n(n+1)=a^{n+2}-\sum\limits_{k=0}^{n+1}\left(-1\right)^{k-1}\binom{n+1}{k}a^{n+2-k}=a\left(a^{n+1}-\left(a-1\right)^{n+1}\right)$.

For $a=5$ answer is $A(n+2)=5\left(5^n-4^n\right)$.

Remark.

For sequence $a_n=\alpha a^n+\beta b^n, n\in\mathbb{N}$, using decomposition, we obtain

$$A_n\left(x\right)=\sum_{k=0}^{n}\left(\alpha a\left(a-1\right)^k+\beta b\left(b-1\right)^k\right)\binom{x}{k}, \text{ and}$$

$$A_n(n+1)=\alpha a\left(a^{n+1}-\left(a-1\right)^{n+1}\right)+\beta b\left(b^{n+1}-\left(b-1\right)^{n+1}\right).$$

Remark.

Before moving to the solution of **Problem 2**, we will consider the one natural question:

What will change in our reasoning if the segment of the sequence $a_{\mathbb{N}_*}$ of length $n+1,$ which detrmine a polynomial, will start not with a_0 but with a_n, for given $n \in \mathbb{N}_*$, that is, if the problem is to stand like this:

For given sequence $a_0, a_1, ..., a_n, ...$ let $A_{m,n}(x)$, $n = 0, 1, 2, ...$ be polynomial defined as follows:

(4) $$\begin{cases} A_{m,n}(k) = a_{m+k}, k = 0, 1, 2, ..., n \\ \deg A_{m,n}(x) \le n \end{cases}.$$

Find the value of $A_{m,n}(n+1)$.

The answer is obvious. Virtually nothing, except for the appearance of yet another index in the notation of the generated polynomial and by replacing $(a_0, a_1, ..., a_n, ...)$

everywhere in obtained results on $(a_m, a_{m+1}, ..., a_{m+n}, ...)$, that is

(2') $$A_{m,n}(n+1) = a_{m+n+1} - \Delta^{n+1}(a_m) = a_{m+n+1} - \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} a_{m+n+1-k} = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} a_{m+n+1-k}$$ and

(3') $$A_{m,n}(x) = a_m + \sum_{k=1}^{n} \Delta^k(a_m) \binom{x}{k} = \sum_{k=0}^{n} \Delta^k(a_m) \binom{x}{k}.$$

Instead **Problem 2** we will solve some it's modification in more general form, namely

Let $(f_n)_{n \ge 1}$ be the Fibonacci sequence defined by $f_0 = 0, f_1 = 1, f_{n+1} = f_n + f_{n-1},\ n \in \mathbb{N}$

and $F_{m,n}(x)$ be polynomial such that $\deg F_{m,n}(x) \le n$ and $F_{m,n}(k) = f_{m+k}$, $k = 0, 1, .., n.$

Determine $F_{m,n}(n+1)$.

Solution.

First note that $\Delta(f_n) = f_{n+1} - f_n = f_{n-1}$.Then $\Delta^k(f_n) = f_{n-k}, k \le n.$

But what happens if $k > n$?

To get answer on that question we should extend definition of Fibonacci sequence on negative part of integer numbers, the more that there is no obstacle to do that.

By replacing n in $f_{n+1} = f_n + f_{n-1}$ with $-n$ we obtain

$$f_{-n+1} = f_{-n} + f_{-n-1} \iff \frac{f_{-n+1}}{(-1)^{n+1}} = \frac{f_{-n}}{(-1)^{n+1}} - \frac{f_{-n-1}}{(-1)^{n+1}} = -\frac{f_{-(n-1)}}{(-1)^n} = \frac{f_{-n}}{(-1)^{n+1}} - \frac{f_{-(n+1)}}{(-1)^{n+2}} \iff \frac{f_{-(n+1)}}{(-1)^{n+2}} = \frac{f_{-n}}{(-1)^{n+1}} + \frac{f_{-(n-1)}}{(-1)^n} \iff$$

$$g_{n+1} = g_n + g_{n-1}, n \in \mathbb{N}, \text{ where } g_n := \frac{f_{-n}}{(-1)^{n+1}}.$$

Since $g_0 = 0$ and $g_1 = \dfrac{f_{-1}}{(-1)^{1+1}} = 1$ we conclude that

$$g_n = f_n \iff \frac{f_{-n}}{(-1)^{n+1}} = (-1)^{n+1} \iff f_{-n} = (-1)^{n+1} f_n.$$

Thus, allowing now for m any integer value we can apply Δ^k to f_m for any $k \in \mathbb{N}$ and do not worry about $sign\,(m-k)$.

In accordance with the above Remark formulas **(2)** and **(3)** becomes

$$F_{m,n}(n+1) = f_{m+n+1} - \Delta^{n+1}(f_m) = f_{m+n+1} - f_{m-n-1}$$

$$F_{m,n}(x) = f_m + \sum_{k=1}^{n} \binom{x}{k} \Delta^k(f_m) = \sum_{k=0}^{n} \binom{x}{k} \Delta^k(f_m) = \sum_{k=0}^{n} \binom{x}{k} f_{m-k}.$$

In particular if $m=1$ we get

$$F_{1,n}(n+1) = f_{n+2} - f_{-n} = f_{n+2} - (-1)^{n+1} f_n = f_{n+2} + (-1)^n f_n$$

and

$$F_{1,n}(x) = \sum_{k=0}^{n} \binom{x}{k} f_{1-k} = 1 + \sum_{k=1}^{n} \binom{x}{k} f_{1-k} = 1 + \sum_{k=1}^{n} \binom{x}{k} f_{1-k} =$$
$$1 + \sum_{k=1}^{n} \binom{x}{k} (-1)^k f_{k-1}.$$

Many years ago, one of my students, in that time 13-year-old Lev Bykhovsky, found another, in my opinion a very beautiful, typically olympiadic, solution of the problem. Retaining the idea, I will bring it here, adapting to the current notation and general setting.

Denote the difference $F_{m,n}(x+1) - F_{m,n}(x)$ via $R_{m,n}(x)$.

Since $\deg F_{m,n}(x+1) = \deg F_{m,n}(x) \leq n$ and both polynomials have the same leading coefficient then $\deg R_{m,n}(x) \leq n-1$.

Besides $R_{m,n}(k) = F_{m,n}(k+1) - F_{m,n}(k) = f_{m+k+1} - f_{m+k} = f_{m+k-1} = f_{m-1+k} = F_{m-1,n-1}(k)$ for $k = 0, 1, ..., n-1$.

And since both polynomials $R_{m,n}(x)$ and $F_{m-1,n-1}(x)$ have a degree not exceeding $n-1$ and their values in n points coincide then the polynomials by themselves coincide, that is, $R_{m,n}(x) = F_{m-1,n-1}(x) \iff F_{m,n}(x+1) - F_{m,n}(x) = F_{m-1,n-1}(x)$.

For $x = n \in \mathbb{N}_*$ we obtain

$F_{m,n}(n+1) - F_{m,n}(n) = F_{m-1,n-1}(n) \iff F_{m,n}(n+1) - F_{m-1,n-1}(n) = F_{m,n}(n) = f_{m+n} = f_{m+n+1} - f_{m+n-1} \iff F_{m,n}(n+1) - f_{m+n+1} = F_{m-1,n-1}(n) - f_{m+n-1}$.

Denoting for convenience $C_{m,n} := F_{m,n}(n+1) - f_{m+n+1}$ we obtain

$$C_{m,n} = C_{m-1,n-1} = ... = C_{m-n,0} \iff F_{m,n}(n+1) - f_{m+n+1} =$$
$$F_{m-n,0}(1) - f_{m-n+0+1} = f_{m-n} - f_{m-n+1} = -f_{m-n-1} \iff$$
$$F_{m,n}(n+1) = f_{m+n+1} - f_{m-n-1}.$$

Let′s see how works this idea if we consider a sequence defined recursively by $a_{n+1} = pa_n + qa_{n-1}$ $(q \neq 0)$. For respect to polynomial $A_{m,n}(x)$ such that $\deg A_{m,n}(x) \leq n$ and $A_{m,n}(k) = f_{m+k}$, $k = 0, 1, .., n$ the question remains the same, namely determine $A_{m,n}(n+1)$.

As before we will use correspondent capital letter for polynomials generated by $a_m, a_{m+1}, ...,$, namely, we will denote this polinomial via $A_{m,n}(x)$.

Let $R_{m,n}(x) := A_{m,n}(x+1) - A_{m,n}(x)$ then for $k = 0, 1, .., n-1$ we have

$$R_{m,n}(k) = A_{m,n}(k+1) - A_{m,n}(k) = a_{m+k+1} - a_{m+k} =$$
$$(p-1)\, a_{m+k} + qa_{m+k-1} = (p-1)\, A_{m,n-1}(k) + qA_{m-1,n-1}(k).$$

Since polynomials $R_{m,n}(x), (p-1)\ A_{m,n-1}(x) + qA_{m-1,n-1}(x)$ have degree not exceeding $n-1$ and their values in n points coincides then

$A_{m,n}(x+1)-A_{m,n}(x)=R_{m,n}(x)=(p-1)\,A_{m,n-1}(x)+qA_{m-1,n-1}(x)$ for all x.

Then for $x=n\in\mathbb{N}_*$ we have recursive relation for $b_{m,n}:=A_{m,n}(n+1)$:
$b_{m,n}-A_{m,n}(n)=(p-1)\,b_{m,n}+qb_{m-1,n-1} \iff$
$b_{m,n}=(p-1)\,b_{m,n-1}+qb_{m-1,n-1}+a_{m+n}$.

I would not say that the recursion for determining $b_{m,n}$ obtained in this way is convenient in the general case of the linear recurrence relation $a_{n+1}=pa_n+qa_{n-1}$.

As we saw above in the case of $a_n=\alpha a^n+\beta b^n, n\in\mathbb{N}$ $(p=a+b, q=-ab)$ the result was achieved much less effort.

But in the case $p=1$ we can still get $b_{m,n}$ by this way, namely:
$b_{m,n}=qb_{m-1,n-1}+a_{m+n} \iff b_{m,n}=qb_{m-1,n-1}+a_{m+n+1}-qa_{m+n-1} \iff$

$c_{m,n}=qc_{m-1,n-1}$,where $c_{m,n}:=b_{m,n}-a_{m+n+1}$.
Hence, $c_{m,n}=q^{m-n}c_{m-n,0} \iff b_{m,n}-a_{m+n+1}=q^{m-n}\left(b_{m-n,0}-a_{m-n+0+1}\right) \iff$

$A_{m,n}\left(n+1\right)=a_{m+n+1}+q^{m-n}\left(A_{m-n,0}\left(1\right)-a_{m-n+0+1}\right)=a_{m+n+1}+q^{m-n}\left(a_{m-n}-a_{m-n+1}\right)=$
$a_{m+n+1}-q^{m-n+1}a_{m-n-1}$.

Generally speaking, the possibility of applying this approach is equivalent to the possibility to present a polynomial $R_{m,n}\left(x\right)$ of degree not bigger then $n-1$, defined by the values in points $x=0,1,...,n-1$, in the form of a linear combination of polynomials $A_{i,,n-1}\left(x\right)$ with $i\le m$,which is not always possible.

This can be verified by attempting to use this approach for sequences $\left(na^n\ \right)_{n\in\mathbb{N}}, \left(\dfrac{1}{n}\right)_{n\in\mathbb{N}}$.

This does not detract from his selective effectiveness, and besides, it is better to have two way to solve the problem than one.

Consider two examples

Example 1.

Let $a_n=na^n$ and $A_n\left(x\right):=A_{1,n}\left(x\right)$ defined by $\begin{cases} A_{1,n}\left(k\right)=a_{k+1}, k=0,1,2,...,n \\ \deg A_{1,n}\left(x\right)\le n \end{cases}$.

Find $A_n\left(n+1\right)$

Since $\Delta^n\left(a_1\right)=\sum\limits_{k=0}^{n}\left(-1\right)^k\dbinom{n}{k}a_{n+1-k}=$
$\sum\limits_{k=0}^{n}\left(-1\right)^k\dbinom{n}{k}\left(n+1-k\right)a^{n+1-k}=\left(n+1\right)\sum\limits_{k=0}^{n}\left(-1\right)^k\dbinom{n}{k}a^{n+1-k}+$
$\sum\limits_{k=1}^{n}\left(-1\right)^k\dbinom{n}{k}ka^{n+1-k}=\left(n+1\right)a\left(a-1\right)^n+an\sum\limits_{k=1}^{n}\left(-1\right)^k\dbinom{n-1}{k-1}a^{n-k}=$
$\left(n+1\right)a\left(a-1\right)^n-an\sum\limits_{k=0}^{n-1}\left(-1\right)^k\dbinom{n-1}{k}a^{n-1-k}=\left(n+1\right)a\left(a-1\right)^n-an\left(a-1\right)^{n-1}$

then $A_n\left(n+1\right)=a_{1+n+1}-\Delta^{n+1}\left(a_1\right)=\left(n+2\right)a^{n+2}-\left(n+2\right)a\left(a-1\right)^{n+1}+a\left(n+1\right)\left(a-1\right)^n=$
$\left(n+2\right)a^{n+2}-a\left(a-1\right)^n\left(\left(n+2\right)a\left(a-1\right)-\left(n+1\right)\right)$.

And now try to use the second way

Let $R_n\left(x\right):=A_n(x+1)-A_n(x)$ then for $k=0,1,...,n-1$ we have

$R_n(k) := A_n(k+1) - A_n(k) = \Delta(a_{k+1}) = (k+2)\,a^{k+2} - (k+1)\,a^{k+1} = a^{k+1}(2a - 1 + k(a-1))$.

Nothing useful.

2. Let $a_n = \dfrac{1}{n}$ and $A_n(x) := A_{1,n}(x)$ defined by $\begin{cases} A_{1,n}(k) = a_{k+1}, k = 0,1,2,...,n \\ \deg A_{1,n}(x) \le n \end{cases}$.

Find $A_n(n+1)$ and $A_n(x)$.

For any natural n we have $\Delta^n(a_1) = \sum\limits_{k=0}^{n} (-1)^k \dbinom{n}{k} a_{n+1-k} = \sum\limits_{k=0}^{n} (-1)^k \dbinom{n}{k} \dfrac{1}{n+1-k} = \dfrac{1}{n+1} + \sum\limits_{k=1}^{n} (-1)^k \dbinom{n}{k} \dfrac{1}{n+1-k} = \dfrac{1}{n+1} + \dfrac{1}{n+1} \sum\limits_{k=1}^{n} (-1)^k \dbinom{n+1}{k}$.

Since $\sum\limits_{k=0}^{n+1} (-1)^k \dbinom{n+1}{k} = 0 \iff 1 + (-1)^{n+1} + \sum\limits_{k=1}^{n} (-1)^k \dbinom{n+1}{k} = 0 \iff$

$\sum\limits_{k=1}^{n} (-1)^k \dbinom{n+1}{k} = -\left(1 + (-1)^{n+1}\right)$ then $\Delta^n(a_1) = \dfrac{1}{n+1} - \dfrac{\left(1 + (-1)^{n+1}\right)}{n+1} = \dfrac{(-1)^n}{n+1}$

and, therefore, $A_n(x) = a_1 + \sum\limits_{k=1}^{n} \Delta^k(a_1) \dbinom{x}{k} = 1 + \sum\limits_{k=1}^{n} \dfrac{(-1)^k}{k+1} \dbinom{x}{k}$

$A_n(n+1) = a_{n+2} - \Delta^{n+1}(a_1) = \dfrac{1}{n+2} - \Delta^{n+1}(a_1) = \dfrac{1}{n+2} - \dfrac{(-1)^{n+1}}{n+2} = \dfrac{1 + (-1)^n}{n+2}$.

Let $a_n = 1$ and $A_{m,n}(x)$ defined by $\begin{cases} A_{m,n}(k) = 1, k = 0,1,2,...,n \\ \deg A_{m,n}(x) \le n \end{cases}$.

Since for the sequence $a_n = 1, n = 0,1,2,...$ we have $\Delta^n(a_1) = 0$ for any natural n and

$A_{m,n}(n+1) = 1, A_{m,n}(x) = 1$ and $\dfrac{k}{k+1} = 1 - \dfrac{1}{k+1}$ then solution of **Problem 3.** immediately follows from solution to example 2.