

Power-sum problem, Bernoulli Numbers and Bernoulli Polynomials.

Arkady M. Alt

Abstract

These notes are the result of attempts to reproduce the path leading from the problem of summation of equal powers of consecutive natural numbers (Power sum problem) to the Bernoulli Numbers and Polynomials.

0.1 Introduction.

This article can be considered as continuation of articles [1] and [2]. I do not claim that represented below way coincide with original. And it is not so important. The important thing is that now for me Bernoulli numbers and polynomials acquired the status of natural born. So, let's start with the Power-sum problem, that is with the following problem:

So, let's start with the Power-sum problem, that is with the following problem:

Find the sum $S_p(n) := 1^p + 2^p + \dots + n^p$ where $p, n \in \mathbb{N}$ (or, using sum notation, $S_p(n) = \sum_{k=1}^n k^p$) in the close form.

Recurrence for $S_p(n)$

Exercise 1 Using representations $1 = (k+1) - k$, $2k = k(k+1) - (k-1)k$, $3k = k(k+1)(k+2) - (k-1)k(k+1)$

find $S_p(n)$ for $p = 1, 2, 3$ and $n \in \mathbb{N}$.

Exercise 2 By summing differences $k^2 - (k-1)^2 = 2k-1$, $k^3 - (k-1)^3 = 3k^2 - 3k + 1$,

$k^4 - (k-1)^4 = 4k^3 - 6k^2 + 4k - 1$ for k running from 1 to n find $S_p(n)$ for $p = 2, 3, 4$.

General case

Exercise 3 For any $p \in \mathbb{N}$ by summing differences $(k+1)^{p+1} - k^{p+1} = \sum_{i=1}^{p+1} \binom{p+1}{i} k^{p+1-i}$ for k running from 1 to n prove that

$$S_p(n) = \frac{(n+1)^{p+1} - n - 1 - \sum_{i=1}^{p-1} \binom{p+1}{i} S_i(n)}{p+1} \quad (1)$$

Exercise 4 For any $p \in \mathbb{N}$ by summing differences $k^{p+1} - k - 1^{p+1} = \sum_{i=1}^{p+1} \binom{p+1}{i} k^{p+1-i}$ for k running from 1 to n prove that

$$S_p(n) = \frac{1}{p+1} \left(n^{p+1} + \sum_{i=1}^p (-1)^{i+1} \binom{p+1}{i+1} S_{p-i}(n) \right) \quad (2)$$

Recurrences (1) and (2) give opportunity, starting from $S_0(n) = \sum_{k=1}^n k^0 = n$, constructively find representation of $S_p(n)$ as polynomial of n .

Since any polynomial degree of m uniquely defined by their values in $m+1$ distinct points ((1) or (2) holds for any natural n) then by such

way polynomials $S_p(x)$ are defined for any $x \in \mathbb{R}$ and $p \in \mathbb{N}$, more precisely, defined sequence of polynomials $(S_p(x))_{p \in \mathbb{N}}$ by recurrence

$$S_p(x) = \frac{(x+1)^{p+1} - 1 - \sum_{i=0}^{p-1} \binom{p+1}{i} S_i(x)}{p+1} \quad (1')$$

(or by recurrence

$$S_p(x) = \frac{1}{p+1} \left(x^{p+1} + \sum_{i=1}^p (-1)^{i+1} \binom{p+1}{i+1} S_{p-i}(x) \right) \quad (2')$$

with initial condition $S_0(x) = x$.

0.2 Bernoulli Numbers and Bernoulli Polynomials

Our goal (as goal for Jacob Bernoulli in his time) solve this recurrence in close form, that is find regular polynomial representation of $S_p(x)$ by powers of x (as linear combination of monomials $1, x, x^2, \dots$.)

Since $S_p(0) = 0$ for any $p = 0, 1, 2, \dots$ (because sum by empty set of indexes, by definition, equal zero) then we should find real numbers s_1, \dots, s_{p+1} such that $S_p(x) = s_1x + \dots + s_{p+1}x^{p+1}$.

Note that the problem would simply be solved if we had known for some polynomial $H(x)$ of degree $p+1$ such that $H(x+1) - H(x) = cx^p$ where c is some constant.

$$\text{Then } S_p(n) = \sum_{k=1}^n k^n = \frac{1}{c} \sum_{k=1}^n (H(k+1) - H(k)) = \frac{H(n+1) - H(1)}{c}.$$

In a sense, one such polynomial (up to arbitrary constant c) we already have $H(x) = S_p(x-1) + c$ since $H(x+1) - H(x) = S_p(x) - S_p(x-1) = x^p$ (we say "we have" in sense of recursive definition).

But our problem just is that $S_p(x)$ has not yet the regular (in powers of x) representation.

Since by differentiation of $S_{p+1}(x) - S_{p+1}(x-1) = x^{p+1}$ we obtain $S'_{p+1}(x) - S'_{p+1}(x-1) = (p+1)x^p$ then $S'_{p+1}(x-1)$ can be considered as another candidate for the role of $H(x)$,

which seems not better than $S_p(x-1)$ by the same reason.

But let's not jump to conclusions and consider in more detail the situation with polynomial $S'_{p+1}(x-1)$ and let's do experiment with the first value of p , that is, see what additional information we can get

applying operation of differentiation to $S_p(x)$.

$$\text{We already know that } S_0(x) = x, S_1(x) = \frac{x(x+1)}{2}, S_2(x) = \frac{x(x+1)(2x+1)}{6}, S_3(x) = \frac{x^2(x+1)^2}{4}$$

Applying recurrences **(1)** or **(2)** we consecutively obtain

$$S_4(x) = \frac{x(x+1)(2x+1)(3x^2+3x-1)}{30} \text{ and } S_5(x) = \frac{x^2(x+1)^2(2x^2+2x-1)}{12}.$$

Accordingly we obtain $S'_0(x) = 1, S'_1(x) = x + \frac{1}{2}, S'_2(x) = x^2 + x +$

$$\frac{1}{6}, S'_3(x) = x^3 + \frac{3x^2}{2} + \frac{x}{2},$$

$$S'_4(x) = x^4 + 2x^3 + x^2 - \frac{1}{30}, S'_5(x) = x^5 + \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{x}{6}$$

$$S''_0(x) = 0, S''_1(x) = 1, S''_2(x) = 2x + 1 = 2\left(x + \frac{1}{2}\right) = 2S'_1(x),$$

$$S''_3(x) = 3x^2 + 3x + \frac{1}{2} = 3\left(x^2 + x + \frac{1}{6}\right) = 3S'_2(x),$$

$$S''_4(x) = 4x^3 + 6x^2 + 2x = 4\left(x^3 + \frac{3x^2}{2} + \frac{x}{2}\right) = 4S'_3(x),$$

$$S''_5(x) = 5x^4 + 10x^3 + 5x^2 - \frac{1}{6} = 5\left(x^4 + 2x^3 + x^2 - \frac{1}{30}\right) = 5S'_4(x).$$

The experiment what we done leads to the assumption that correlation $S''_p(x) = pS'_{p-1}(x)$ holds for any $p \in \mathbb{N}$.

In fact, assuming $S''_i(x) = pS'_{i-1}(x), i = 1, 2, \dots, p-1$, and by differentiating **(1')** twice we obtain

$$S'_p(x) = \frac{(p+1)(x+1)^p - \sum_{i=0}^{p-1} \binom{p+1}{i} S'_i(x)}{p+1} = \frac{(p+1)(x+1)^p - 1 - \sum_{i=1}^{p-1} \binom{p+1}{i} S'_i(x)}{p+1}$$

and

$$\begin{aligned}
S_p''(x) &= \frac{(p+1)p(x+1)^{p-1} - \sum_{i=1}^{p-1} \binom{p+1}{i} S_i''(x)}{p+1} = \\
&= \frac{(p+1)p(x+1)^{p-1} - \sum_{i=1}^{p-1} \binom{p+1}{i} i S_{i-1}'(x)}{p+1} = \\
&= \frac{(p+1)p(x+1)^{p-1} - (p+1) \sum_{i=1}^{p-1} \binom{p}{i-1} S_{i-1}'(x)}{p+1} = \\
p S_{p-1}'(x) &= p \cdot \frac{(x+1)^{p-1} - \sum_{i=0}^{p-2} \binom{p}{i} S_{i-1}'(x)}{p} = \\
&= p S_{p-1}'(x).
\end{aligned}$$

Exercise 5 Prove $S_p''(x) = p S_{p-1}'(x)$ for any $p \in \mathbb{N}$ using (2').

Thus with Math Induction proved that $S_p''(x) = p S_{p-1}'(x)$ for any $p \in \mathbb{N}$.

Coming back to polynomial $S_p'(x-1)$ we denote it by $B_p(x)$ and then by replacing x with $x-1$ in the recurrence

$$S_p'(x) = \frac{(p+1)(x+1)^p - \sum_{i=0}^{p-1} \binom{p+1}{i} S_i'(x)}{p+1}$$

we obtain the following recurrence for polynomials $B_p(x), p \in \mathbb{N}$:

$$B_p(x) = (x-1)^p + \frac{\sum_{i=1}^p (-1)^{i+1} \binom{p+1}{i+1} B_{p-i}(x)}{p+1}. \quad (\text{B2})$$

So, what we know at the moment of the polynomial $B_p(x)$?

0.3 Properties.

P0. $\deg B_p(x) = \deg S_p'(x-1) = p$;

P1. $B_0(x) = S_1'(x-1) = 1$;

P2. $B_p'(x) = (S_p'(x-1))' = S_p''(x-1) = p S_{p-1}'(x) = p B_{p-1}(x)$;

P3. $B_p(x+1) - B_p(x) = S_p'(x) - S_p'(x-1) = p x^{p-1}, p \in \mathbb{N}$.

Such defined polynomials we will call *Bernoulli Polynomials*.

By the way, in fact, we already have the first few polynomials $B_p(x)$, namely,

$$\begin{aligned}
B_1(x) &= S_1'(x-1) = x-1 + \frac{1}{2} = x - \frac{1}{2}, B_2(x) = S_2'(x-1) = \\
(x-1)^2 + (x-1) + \frac{1}{6} &= x^2 - x + \frac{1}{6}, \\
B_3(x) &= S_3'(x-1) = (x-1)^3 + \frac{3(x-1)^2}{2} + \frac{(x-1)}{2} = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x;
\end{aligned}$$

$$B_4(x) = S'_4(x-1) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \quad B_5(x) = S'_5(x-1) = x^5 - \frac{5x^4}{2} + \frac{5x^3}{3} - \frac{x}{6}.$$

We can see that $B_1(0) = -\frac{1}{2}, B_1(1) = \frac{1}{2}$, but $B_2(0) = B_2(1) = \frac{1}{6}, B_3(0) = B_3(1) = 0, B_4(0) = B_4(1) = -\frac{1}{30}, B_5(0) = B_5(1) = 0$ and in general $B_p(0) = B_p(1)$ for any $p \geq 2$. Furthermore, $B_{2p+1}(0) = B_{2p+1}(1) = 0$.

Indeed, since $B_p(x+1) - B_p(x) = px^{p-1}$ then for $x=0$ we obtain $B_p(1) - B_p(0) = p \cdot 0^{p-1} \iff B_p(1) = B_p(0)$ for all $p \geq 2$.

(Hypothesis $B_{2p+1}(0) = B_{2p+1}(1) = 0, p \in \mathbb{N}$ is equivalent to divisibility $B_{2p+1}(x)$ by x and will be proved later).

Note that recursion $B'_p(x) = pB_{p-1}(x), p \in \mathbb{N}$ with initial condition $B_0(x) = 1$ allows to get consistently polynomials $B_1(x), B_2(x), B_3(x), \dots$ and thus easier than by recurrence **(B1)** or **(B2)**.

Indeed, assume that we already know polynomial $B_{p-1}(x)$, then $B_p(x) - B_p(1) = \int_1^x B'_p(t) dt = \int_1^x pB_{p-1}(t) dt \iff B_p(x) = B_p(1) + p \int_1^x B_{p-1}(t) dt$.

Let $B_p := B_p(0), p \in \mathbb{N} \cup \{0\}$. Such defined numbers we call *Bernoulli Numbers*.

By replacing x in **(B1)** or in **(B2)** with 0 we obtain

$$B_p = \frac{-\sum_{i=0}^{p-1} \binom{p+1}{i} B_i}{p+1} \quad \text{(B3)}$$

or

$$B_p = (-1)^p + \frac{\sum_{i=1}^p (-1)^{i+1} \binom{p+1}{i+1} B_{p-i}}{p+1}. \quad \text{(B4)}$$

Any of these recurrences allows to get consistently numbers B_1, B_2, B_3, \dots

Exercise 6 Find first 5 terms of sequence $(B_p)_{p \geq 0}$.

We show, how knowing $B_k, k = 1, 2, \dots$ we can obtain polynomial $B_p(x)$.

Let $B_p(x) = b_px^p + b_{p-1}x^{p-1} + \dots + b_1x + b_0$, where b_k should be determined.

Since $B_p(0) = B_p$ then $b_0 = B_p$. Also since $B'_p(x) = pB_{p-1}(x)$ then $B_p^{(k)}(x) = p(p-1)\dots(p-k+1)B_{p-k}(x)$ and

$B_p^{(k)}(x) = (b_px^p + b_{p-1}x^{p-1} + \dots + b_1x + b_0)^{(k)} = (b_px^p + b_{p-1}x^{p-1} + \dots + b_{k+1}x^k)^{(k)} + b_k k!$ yields

$$B_p^{(k)}(0) = b_k k! \iff p(p-1)\dots(p-k+1)B_{p-k}(0) = b_k k! \iff b_k = \frac{p(p-1)\dots(p-k+1)}{k!} B_{p-k} \iff$$

$$b_k = \binom{p}{k} B_{p-k}, k = 1, 2, \dots, p.$$

$$\text{Hence, } B_p(x) = B_p + \binom{p}{1} B_{p-1} x^1 + \dots + \binom{p}{p-1} B_1 x^{p-1} + B_0 x^p = \sum_{k=0}^p \binom{p}{k} B_{p-k} x^k.$$

$$\text{In particular } B_0(x) = x, B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6},$$

$$B_3(x) = B_0 x^3 + 3B_1 x^2 + 3B_2 x + B_3 = x^3 + 3 \left(-\frac{1}{2} \right) x^2 + 3 \cdot \frac{1}{6} x = x^3 - \frac{3}{2} x^2 + \frac{1}{2} x.$$

More properties of Bernoulli polynomials and numbers.

P4. $\int_0^1 B_p(x) dx = 0$ for any $p \in \mathbb{N}$.

Proof. Since by **P2**. we have $B'_{p+1}(x) = (p+1) B_p(x)$ then

$$(p+1) \int_0^1 B_p(x) dx = (p+1) \int_0^1 B'_{p+1}(x) dx = (p+1) (B_{p+1}(x))_0^1 = (p+1) (B_{p+1}(1) - B_{p+1}(0)) = (p+1) \cdot 0 = 0 \implies \int_0^1 B_p(x) dx = 0.$$

We will prove that properties **P1.**, **P2.**, **P3.** determine polynomials $B_p(x)$ uniquely.

Let $(Q_p(x))_{p \geq 0}$ be sequence of polynomials such that $Q_0(x) = 1, Q'_n(x) = nQ_{n-1}(x), n \in \mathbb{N}$ and $Q_p(x+1) - Q_p(x) = px^{p-1}, p \in \mathbb{N}$.

First note that $Q_0(x) = 1 = B_0(x)$. Also note that $Q_n(1) = Q_n(0)$ for $n \geq 2$ since $Q_p(1) - Q_p(0) = p \cdot 0^{p-1} = 0, p \geq 2$.

This yields $\int_0^1 Q_p(x) dx = 0, p \in \mathbb{N}$.

Indeed, $p \int_0^1 Q_p(x) dx = \int_0^1 Q'_{p+1}(x) dx = Q_{n+1}(1) - Q_{n+1}(0) = 0$. Since $Q'_1(x) = 1 \cdot Q_0(x) = 1$ then $Q_1(x) = x + c$ and, therefore, $Q'_2(x) = 2Q_1(x)$

yields $Q_2(x) = x^2 + 2cx + d$. Then $Q_2(x+1) - Q_2(x) = 2x \iff (x+1)^2 + 2c(x+1) - x^2 - 2cx = 2x \iff$

$$2c + 1 = 0 \iff c = -\frac{1}{2}. \text{ Hence, } Q_1(x) = x - \frac{1}{2} = B_1(x)$$

Assume that $Q_p(x) = B_p(x)$ then $Q'_{p+1}(x) = (p+1)Q_p(x) = (p+1)B_p(x) = B'_{p+1}(x) \iff$

$$Q_{p+1}(x) = B_{p+1}(x) + c. \text{ Therefore } 0 = \int_0^1 Q_{p+1}(x) dx = \int_0^1 (B_{p+1}(x) + c) dx = \int_0^1 B_{p+1}(x) dx + c = c.$$

So, by Math Induction $Q_p(x) = B_p(x)$ for any $p \in \mathbb{N}$. ■

P5. $B_p(x) = (-1)^p B_p(1-x), p \geq 0$. (Complement property)

Proof. Let $Q_p(x) := (-1)^p B_p(1-x), p \in \mathbb{N} \cup \{0\}$ then:

1. By **P1** $Q_0(x) = B_0(1-x) = 1$;

2. By **P2**. $Q'_p(x) = (-1)^p (B_p(1-x))' = (-1)^p (B_p(1-x))' = -(-1)^p B'_p(1-x) = p(-1)^{p-1} B_{p-1}(1-x) = pQ_{p-1}(x)$;

3. By **P3**. $Q_p(x+1) - Q_p(x) = (-1)^p B_p(1-(x+1)) - (-1)^p B_p(1-x) =$

$$(-1)^p (B_p(-x) - B_p(1 + (-x))) = (-1)^{p+1} (B_p((-x) + 1) - B_p(-x)) = p(-1)^{p+1} (-x)^{p-1} = px^{p-1}.$$

Therefore, by property of uniqueness we get $(-1)^p B_p(1 - x) = B_p(x)$.

■

Corollary 7 For $p = 2m + 1, m \in \mathbb{N}$ holds $B_p(0) = 0$.

Indeed, if $p = 2m + 1$ then $B_p(x) = -B_p(1 - x)$ and, therefore, for $x = 0$ we have $B_p(0) = -B_p(1) = -B_p(0) \implies 2B_p(0) = 0 \iff B_p(0) = 0$.

Corollary 8 By replacing x in $B_p(x) = (-1)^p B_p(1 - x)$ with $x + 1$ we obtain

$$B_p(x + 1) = (-1)^p B_p(1 - (x + 1)) = (-1)^p B_p(-x) = (-1)^p \sum_{k=0}^p \binom{p}{k} B_{p-k}(-x)^k = \sum_{k=0}^p (-1)^{n-k} \binom{p}{k} B_{p-k}x^k = \sum_{k=0}^p (-1)^k \binom{p}{k} B_{p-k}x^k.$$

Now we ready write $S_p(n)$ in the form of polynomial by powers of

n .

$$\text{Since } B_{p+1}(x + 1) - B_{p+1}(x) = (p + 1)x^p \text{ then } (p + 1)S_p(n) = (p + 1) \sum_{k=1}^n k^p =$$

$$\sum_{k=1}^n (B_{p+1}(k + 1) - B_{p+1}(k)) = B_{p+1}(n + 1) - B_{p+1}(1) = B_{p+1}(n + 1) -$$

$B_{p+1}(0)$

and, therefore, $(p + 1)S_p(n) = B_{p+1}(n + 1) - B_{p+1} \iff$

$$S_p(n) = \frac{B_{p+1}(n + 1) - B_{p+1}}{p + 1} = \frac{1}{p + 1} \left(\sum_{k=0}^{p+1} (-1)^k \binom{p + 1}{k} B_{p+1-k}n^k - B_{p+1} \right) =$$

$$\frac{1}{p + 1} \sum_{k=1}^{p+1} (-1)^k \binom{p + 1}{k} B_{p+1-k}n^k.$$

$$(\star) \quad S_p(n) = \frac{1}{p + 1} \sum_{k=1}^{p+1} (-1)^k \binom{p + 1}{k} B_{p+1-k}n^k. \text{(Faulhaber's$$

Formula).

Problem 1

Prove that $B_{2m+1}(x)$ divisible by $S_2(x - 1)$ for any $m \in \mathbb{N}$.

Problem 2

Prove that $Sign(B_{2m}) = (-1)^{m+1}$ and $\max_{[0,1]} B_{4m-2}(x) = B_{4m-2}, \min_{[0,1]} B_{4m}(x) =$

$B_{4m}, m \in \mathbb{N}$.

Hint (use Math Induction).

1. A. M. Alt-Variations on a theme-The sum of equal powers of natural numbers,

part 1_Crux vol.40,n.8;

2. A. M. Alt-Variations on a theme-The sum of equal powers
of natural numbers,
part 2 _ Crux vol.40,n.10.