

About one inequality from APMO, 2004 (New solution and generalizations).

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Original problem from **APMO, 2004/5** is:

Prove that

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 9(ab + bc + ca)^2$$

for any positive real numbers a, b, c .

1. We will prove more stronger inequality

$$(1) \quad (a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 3(a + b + c)^2$$

which yields original inequality (because $(a + b + c)^2 \geq 3(ab + bc + ca)$).

Since for any real x, y holds inequalities

$$(2) \quad (x^2 + 2)(y^2 + 2) \geq 2(x + y)^2 \iff (xy - 2)^2 \geq 0 \text{ and}$$

$$(3) \quad (x^2 + 2)(y^2 + 2) \geq \frac{3}{2}((x + y)^2 + 2) \iff 2(xy - 1)^2 + (x - y)^2 \geq 0$$

then

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq \frac{3}{2}((a + b)^2 + 2)(c^2 + 2) \geq \frac{3}{2} \cdot 2((a + b) + c)^2 = 3(a + b + c)^2$$

with equality condition $a = b = c$, which follows immediately from equality condition for **(2)** and **(3)**.

2. Generalization 1.

For any positive real numbers x, y, z, t holds inequality

$$(4) \quad (x^2 + t^2)(y^2 + t^2)(z^2 + t^2) \geq \frac{3t^4}{4}(x + y + z)^2.$$

Proof.

Since for any real x, y and t holds inequalities

$$(5) \quad (x^2 + t^2)(y^2 + t^2) \geq t^2(x + y)^2 \iff (xy - t^2)^2 \geq 0$$

and

(6)

$$(x^2 + t^2)(y^2 + t^2) \geq \frac{3t^2}{4}((x + y)^2 + t^2) \iff \left(xy - \frac{t^2}{2}\right)^2 + \frac{t^2}{4}(x - y)^2 \geq 0$$

then

$$(x^2 + t^2)(y^2 + t^2)(z^2 + t^2) \geq \frac{3t^2}{4} \left((x+y)^2 + t^2 \right) (z^2 + t^2) \geq \frac{3t^2}{4} \cdot t^2 ((x+y) + z)^2 = \frac{3t^4}{4} (x+y+z)^2$$

with equality condition $x = y = z = \frac{t}{\sqrt{2}}$, which follows

immediately from equality condition for (5) and (6).

Those inequalities can be obtained from the similar inequalities in the

particular case $a = \sqrt{2}$ by substitution $a = \frac{\sqrt{2}x}{t}$, $b = \frac{\sqrt{2}y}{t}$, $c = \frac{\sqrt{2}z}{t}$.

3. Generalization 2.

Let $x, y, t > 0$ then

$$(7) \quad (x^n + t^n)(y^n + t^n) \geq \frac{t^n}{2^{n-2}} (x+y)^n, n \geq 2.$$

Indeed, since by Power Mean Inequality

$$\begin{aligned} \left(\frac{y^n + t^n}{2} \right)^{\frac{1}{n}} &\geq \left(\frac{y^{\frac{n}{n-1}} + t^{\frac{n}{n-1}}}{2} \right)^{\frac{n-1}{n}} \iff \\ \frac{y^n + t^n}{2} &\geq \left(\frac{y^{\frac{n}{n-1}} + t^{\frac{n}{n-1}}}{2} \right)^{n-1} \iff \\ y^n + t^n &\geq \frac{\left(y^{\frac{n}{n-1}} + t^{\frac{n}{n-1}} \right)^{n-1}}{2^{n-2}} \quad \text{and by Holder's Inequality} \end{aligned}$$

$$(x^n + t^n) \left(y^{\frac{n}{n-1}} + t^{\frac{n}{n-1}} \right)^{n-1} = (x^n + t^n) \left(t^{\frac{n}{n-1}} + y^{\frac{n}{n-1}} \right)^{n-1} \geq (xt + ty)^n = t^n (x+y)^n$$

$$\text{then } (x^n + t^n)(y^n + t^n) \geq \frac{1}{2^{n-2}} (x^n + t^n) \left(y^{\frac{n}{n-1}} + t^{\frac{n}{n-1}} \right)^{n-1} \geq \frac{t^n}{2^{n-2}} (x+y)^n.$$

4. Generalization 3.

For any natural m and n such that $2 \leq m \leq n$ and positive real numbers

x_1, x_2, \dots, x_m, t holds inequality

$$(8) \quad \prod_{k=1}^m (x_k^n + t^n) \geq \frac{t^{n(m-1)}}{m^{n-m} (m-1)^{m-1}} \cdot (x_1 + x_2 + \dots + x_m)^n$$

with equality condition $x_k = \frac{t}{\sqrt[n]{m-1}}$, $k = 1, 2, \dots, m$.

Proof.

Denote $P := \prod_{k=1}^m (x_k^n + t^n)$ and $S := \sum_{k=1}^m \frac{1}{x_k^n + t^n}$ then by AM-GM Inequality

$$\begin{aligned} \text{we have} \\ \frac{x_i^n}{x_i^n + t^n} + \sum_{k=1, k \neq i}^m \frac{t^n}{x_k^n + t^n} \cdot \frac{1}{m-1} &= \frac{x_i^n}{x_i^n + t^n} - \frac{t^n}{(m-1)(x_i^n + t^n)} + \frac{St^n}{m-1} \geq \\ m \sqrt[m]{\frac{x_i^n t^{n(m-1)}}{P(m-1)^{m-1}}}, i &= 1, 2, \dots, m. \end{aligned}$$

Summing this inequalities we obtain:

$$\sum_{i=1}^m \frac{x_i^n}{x_i^n + t^n} + \frac{mt^n S}{m-1} - \frac{t^n}{m-1} \sum_{i=1}^m \frac{1}{x_i^n + t^n} = \sum_{i=1}^m \frac{x_i^n}{x_i^n + t^n} + t^n S =$$

$$\sum_{i=1}^m \frac{x_i^n}{x_i^n + t^n} + \sum_{k=1}^m \frac{t^n}{x_k^n + t^n} = \sum_{i=1}^m \frac{x_i^n + t^n}{x_i^n + t^n} = m \geq$$

$$m \sum_{i=1}^m \sqrt[m]{\frac{x_i^n t^{n(m-1)}}{P(m-1)^{m-1}}} \iff P^{\frac{1}{m}} \geq \frac{t^{\frac{n(m-1)}{m}} \cdot \sum_{i=1}^m x_i^{\frac{n}{m}}}{(m-1)^{\frac{m-1}{m}}} \iff$$

$$P \geq \frac{t^{n(m-1)}}{(m-1)^{m-1}} \left(\sum_{i=1}^m x_i^{\frac{n}{m}} \right)^m = \frac{t^{n(m-1)} m^m}{(m-1)^{m-1}} \left(\frac{\sum_{i=1}^m x_i^{\frac{n}{m}}}{m} \right)^m.$$

Since by Power Mean Inequality

$$\left(\frac{\sum_{i=1}^m x_i^{\frac{n}{m}}}{m} \right)^{\frac{m}{n}} \geq \frac{\sum_{i=1}^m x_i}{m} \iff \left(\sum_{i=1}^m x_i^{\frac{n}{m}} \right)^m \geq \frac{\left(\sum_{i=1}^m x_i \right)^n}{m^{n-m}}$$

then we finally get inequality

$$P \geq \frac{t^{n(m-1)}}{(m-1)^{m-1} m^{n-m}} \left(\sum_{i=1}^m x_i \right)^n,$$

where equality occurs iff $x_1 = x_2 = \dots = x_m = \frac{t}{\sqrt[n]{m-1}}$
(condition of equality in Power Mean Inequality).

Applications.

Problem 3326(a). (CRUX, Vol.34, No.3)

Let $a, b,$ and c be positive real numbers.

(a) Show that $\prod_{cyc} (a^2 + 2) + 4 \prod_{cyc} (a^2 + 1) \geq 6(a + b + c)^2$.

Solution.

Applying inequality (4) to $t = \sqrt{2}$ and $t = 1$ we obtain

$$\prod_{cyc} (a^2 + 2) \geq \frac{3 \cdot (\sqrt{2})^4}{4} (a + b + c)^2 = 3(a + b + c)^2 \text{ and}$$

$$\prod_{cyc} (a^2 + 1) \geq \frac{3 \cdot 1^4}{4} (a + b + c)^2 = \frac{3}{4} (a + b + c)^2.$$

Hence, $\prod_{cyc} (a^2 + 2) + 4 \prod_{cyc} (a^2 + 1) \geq 6(a + b + c)^2$.

Problem 3327(a). (CRUX, Vol.34, No.3)

Let $a, b,$ and c be positive real numbers.

(a) Show that $\prod_{cyc} (a^4 + 3a^2 + 2) \geq \frac{9}{4} (a + b + c)^2$.

Solution.

$$\prod_{cyc} (a^4 + 3a^2 + 2) = \prod_{cyc} (a^2 + 2) \cdot \prod_{cyc} (a^2 + 1) \geq 3(a + b + c)^2 \cdot \frac{3}{4} (a + b + c)^2 =$$

$$\frac{9}{4}(a+b+c)^2.$$

Remark.

There are many solutions of this APMO **2004** problem 5, but all what I saw are more complicated and heavy.

For example: 3 solutions of Problem 67 in "Old and New Inequalities" Titu Andreescu and Vasile Cirtoaje, and 2 solutions in "Topics in Inequalities" of Hojoo Lee and also solution in

<https://mks.mff.cuni.cz/kalva/apmo/asoln/asol045.html>