$$\lim_{n\to\infty}\frac{1}{n+2}((n+2)!)^{\frac{2n+3}{(n+2)^2}}(sf(n+1))^{\frac{2n+3}{(n+2)^2}-\frac{2n+1}{(n+1)^2}}=$$

$$\lim_{n\to\infty}\frac{1}{n+2}e^{((n+2)\ln(n+2)-n-2)\left(\frac{2}{n}+O(\frac{1}{n^2})\right)}\left(\sqrt{2\pi(n+2)}(1+o(1))\right)^{\frac{2}{n}+O(\frac{1}{n^2})}\cdot$$

$$\cdot\frac{1}{n}\frac{n}{(sf(n+1))^{\frac{2}{n^2}}\left(1+O(\frac{1}{n})\right)}=$$

$$=\lim_{n\to\infty}\frac{1}{n(n+2)}e^{((n+2)\ln(n+2)-n-2)\frac{2}{n}}\frac{n}{(sf(n+1))^{\frac{2}{n^2}}}=$$

$$=\lim_{n\to\infty}\frac{1}{n(n+2)}e^{2\ln n-2+O(\frac{\ln n}{n})}e^{\frac{3}{2}}=e^{-2+\frac{3}{2}}$$

Finally

$$\lim_{n\to\infty}e^{\frac{3}{2}}\left(\frac{\sqrt[n^2]{sf(n)}}{\sqrt[(n+1)^2]{sf(n+1)}}\right)^n=\lim_{n\to\infty}e^{\frac{3}{2}}e^{-2+\frac{3}{2}}=e$$

and then

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt[n^2]{sf(n)}}\frac{\left(\frac{\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}}}{\frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}}-1\right)}{\ln\left(\frac{\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}}}{\frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}}\right)}\ln\left(\frac{\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}}}{\frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}}\right)^n=e^{\frac{3}{4}}\cdot 1\cdot 1=e^{\frac{3}{4}}$$

and this concludes the proof.

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W9. (Solution by the proposer.) Consider recurrence

$$x_{n+1}=\frac{1}{k}\left(x_n+x_{n-1}+x_{n-2}+x_{n-k+1}\right),n\geq k-1$$

with $x_n\in\mathbb{C},n\in\mathbb{N}\cup\{0\}$.
Easy to see that the sequence $(x_n)_{n\geq 0}$, originally defined by linear homogeneous recurrence of k-degree can be equivalently defined by the following linear nonhomogeneous recurrence of degree $k-1$

$$x_{n+1}+\frac{k-1}{k}x_n+\frac{k-2}{k}x_{n-1}+...+\frac{1}{k}x_{n-k+2}=$$

$$=\frac{kx_{k-1}+(k-1)\,x_{k-2}+...+2x_1+x_0}{k},n\geq k-2$$

which further, by substitution $x_n=t_n+\delta$ – where

$$\delta=\frac{kx_{k-1}+(k-1)\,x_{k-2}+...+2x_1+x_0}{\binom{k}{2}},$$

can be reduced to recurrence

$$t_{n+1}+\frac{k-1}{k}t_n+\frac{k-2}{k}t_{n-1}+...+\frac{1}{k}t_{n-k+2}=0 \iff$$

$$kt_{n+1}+(k-1)\,t_n+(k-2)\,t_{n-1}+...+t_{n-k+2}=0,\ n\geq k-2. \tag{1}$$

Characteristic polynomial for **(1)** is
$kz^{k-1}+(k-1)\,z^{k-2}+...+2z+1=P'(z)$ where
$P(z)=z^k+z^{k-1}+....+z+1.$
By *Gauss–Lucas theorem* for any polynomial $Q(z)$ all roots of Q' lie within the convex hull of the roots of Q, that is within the smallest convex polygon containing the roots of P. (Everywhere further we will use notation $e^{i\alpha}$ instead $\cos\alpha+i\sin\alpha$)
Since

$$P(z)=0 \iff \begin{cases} z^{k+1}=1 \\ z\neq 1 \end{cases} \iff z\in\left\{e^{\frac{2j\pi}{k+1}}\mid j=1,2,...,k\right\}$$

then all roots of P are simple and lie on unite circle $|z|=1$ (vertices of regular polygon, inscribed in circle $|z|=1$).
Therefore, for any root z of equation $P'(z)=0$ holds inequality $|z|<1$.
Indeed, since $z=\sum_{j=1}^{k}p_je^{\frac{2j\pi}{k+1}}$,where $\sum_{j=1}^{k}p_j=1,p_j\in[0,1)\,,\ j=1,2,...,k$ then

$$|z|=\left|\sum_{j=1}^{k}p_je^{\frac{2j\pi}{k+1}}\right|<\sum_{j=1}^{k}p_j\left|e^{\frac{2j\pi}{k+1}}\right|=\sum_{j=1}^{k}p_j=1.$$

Since $x_n=\delta+\sum_{j=1}^{k}c_jz_j^n$, where $|z_j|<1,j=1,2,...,k$ then

$$\lim_{n\to\infty} x_n = \delta + \sum_{j=1}^{k} c_j \lim_{n\to\infty} z_j^n = \delta.$$

W10. (Solution by the proposer.) First we will prove, using Math. Induction, inequality for $m = n$, namely inequality

$$\left(\frac{a^n + b^n + c^n}{a^{n-1} + b^{n-1} + c^{n-1}}\right)^{n+1} \geq \frac{a^{n+1} + b^{n+1} + c^{n+1}}{3}, \ n \geq 3. \tag{1}$$

1. *Base of Math. Induction.*
For $n = 3$ we have

$$\left(\frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2}\right)^4 \geq \frac{a^4 + b^4 + c^4}{3}. \tag{2}$$

Assume $a + b + c = 1$ (due to homogeneity of (2) and denote $p := ab + bc + ca, q := abc$. Since

$$a^2 + b^2 + c^2 = 1 - 2p, a^3 + b^3 + c^3 = 1 + 3q - 3p, a^4 + b^4 + c^4 =$$

$$= 1 + 4q - 4p + 2p^2$$

inequality (2) becomes

$$\left(\frac{1 + 3q - 3p}{1 - 2p}\right)^4 \geq \frac{1 + 4q - 4p + 2p^2}{3} \iff h(q) \geq \frac{(1 - 2p)^4}{3}$$

where

$$h(q) := \frac{(1 + 3q - 3p)^4}{1 + 4q - 4p + 2p^2}.$$

Noting that $p \leq \frac{1}{3}$ ($\iff ab + bc + ca \leq \frac{(a+b+c)^2}{3}$) and $9q \geq 4p - 1$($\iff \sum_{cyc} a(a-b)(a-c) \geq 0$ –Schure inequality in *1-p-q* notation) we can see that

$$h'(q) = \frac{12(1 + 3q - 3p)^3 (1 + 4q - 4p + 2p^2) - 4(1 + 3q - 3p)^4}{1 + 4q - 4p + 2p^2} =$$

$$= \frac{4(1 + 3q - 3p)^3 (6p^2 - 9p + 2 + 9q)}{1 + 4q - 4p + 2p^2} \geq 0$$