Three Constants.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Let R, r and s be, respectively,circumradius, inradius and semiperimeter of a triangle.

a) Prove inequality

$$R^2-4r^2 \geq \frac{1}{5}\cdot(s^2-27r^2);$$

b) Find the maximum value for constant K such that inequality

$$R^2-4r^2 \geq K(s^2-27r^2)$$

holds for any triangle;

c) Find the $\lim\limits_{R\to 2r}\dfrac{R^2-4r^2}{s^2-27r^2}$.

Solution.

a) Follows immediately from inequality $s^2 \leq 4R^2+4Rr+3r^2$. Really,

$$R^2-4r^2 \geq \frac{1}{5}\cdot(s^2-27r^2) \Leftrightarrow s^2 \leq 5R^2+7r^2 \Leftrightarrow$$

$$0 \leq (4R^2+4Rr+3r^2-s^2)+(R-2r)^2.$$

b) Recall ([1]), that a triple (R,r,s) of positive real numbers can determine a triangle, where $R, r,$ and s be a circumradius, inradius and semiperimeter respectively iff $(R,r,s) \in \bar{\Delta} := \{(R,r,s) \mid R \geq 2r$ and $L(R,r) \leq s^2 \leq M(R,r)\}$where

$$L(R,r) = 2R^2+10Rr-r^2-2(R-2r)\sqrt{R(R-2r)}$$ and

$$M(R,r) = 2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)}.$$

Since a triangle is equilateral iff $R = 2r$ then set

$$\Delta := \left\{(R,r,s) \mid (R,r,s) \in \bar{\Delta} \text{ and } R \neq 2r\right\}$$

determine all non-equilateral triangles.

Thus, $\max K = \min\limits_{(R,r,s)\in\Delta}\dfrac{R^2-4r^2}{s^2-27r^2} = \min\limits_{R>2r}\left(\min\limits_{s}\dfrac{R^2-4r^2}{s^2-27r^2}\right) = \min\limits_{R>2r}\dfrac{R^2-4r^2}{M(R,r)-27r^2} =$

$$\min_{R>2r}\frac{R^2-4r^2}{2R^2+10Rr-28r^2+2(R-2r)\sqrt{R(R-2r)}} = \min_{R>2r}\frac{R+2r}{2R-14r+2\sqrt{R(R-2r)}} =$$

$$\min_{R>2r}\frac{1+\dfrac{2r}{R}}{2-7\cdot\dfrac{2r}{R}+2\sqrt{1-\dfrac{2r}{R}}}.$$

Denoting $t := \sqrt{1-\dfrac{2r}{R}}$ we obtain that $t \in (0,1)$ and, therefore,

$K_* := \max K = \min\limits_{t\in(0,1)}\dfrac{2-t^2}{9+2t-7t^2} = \dfrac{23+\sqrt{17}}{8}$, because $\dfrac{23+\sqrt{17}}{128}$ is smallest real k for which equation $\dfrac{2-t^2}{9+2t-7t^2} = k$ have solution in $(0,1)$.

Indeed, if equation $\dfrac{2-t^2}{9+2t-7t^2} = k$ have solution then

$2-t^2 = k(9+2t-7t^2) \Leftrightarrow (7k-1)t^2-2kt-9k+2 = 0$ yields

$k^2+(7k-1)(9k-2) = 64k^2-23k+2 \geq 0 \Rightarrow k \geq \dfrac{23+\sqrt{17}}{128}$.

Since for $k_* := \dfrac{23+\sqrt{17}}{128}$ equation $\dfrac{2-t^2}{9+2t-7t^2} = k_*$ have only solution

$t_* = \dfrac{k_*}{7k_* - 1} = \dfrac{23 + \sqrt{17}}{33 + 7\sqrt{17}} = \dfrac{5 - \sqrt{17}}{2} \in (0,1)$ then

$K_* = \min\limits_{t \in (0,1)} \dfrac{2 - t^2}{9 + 2t - 7t^2} = \dfrac{2 - t_*^2}{9 + 2t_* - 7t_*^2} = \dfrac{23 + \sqrt{17}}{128}$.

So, for any triangle holds inequality

$$R^2 - 4r^2 \geq \frac{23 + \sqrt{17}}{128}(s^2 - 27r^2)$$

and not exist constant $K > \dfrac{23 + \sqrt{17}}{128}$ which provide inequality in (b).

c) $\lim\limits_{R \to 2r} \dfrac{R^2 - 4r^2}{s^2 - 27r^2} = \dfrac{2}{9}$ because $\dfrac{R^2 - 4r^2}{2R^2 + 10Rr - 28r^2 + 2(R - 2r)\sqrt{R(R - 2r)}} \leq$

$\dfrac{R^2 - 4r^2}{s^2 - 27r^2} \leq \dfrac{R^2 - 4r^2}{2R^2 + 10Rr - 28r^2 - 2(R - 2r)\sqrt{R(R - 2r)}} \Leftrightarrow$

$\dfrac{R + 2r}{2R + 14r + 2\sqrt{R(R - 2r)}} \leq \dfrac{R^2 - 4r^2}{s^2 - 27r^2} \leq \dfrac{R + 2r}{2R + 14r - 2\sqrt{R(R - 2r)}}$ and

$\lim\limits_{R \to 2r} \dfrac{R + 2r}{2R + 14r + 2\sqrt{R(R - 2r)}} = \lim\limits_{R \to 2r} \dfrac{R + 2r}{2R + 14r - 2\sqrt{R(R - 2r)}} = \dfrac{2}{9}$.

[**1**] **D.S.Mitrinovic**,**J.E.Pecaric**,**V.Volnec.Recent Advances In Geometric Inequalities**.