which is the result. The proof is complete.

Paolo Perfetti

W7. (Solution by the proposer.) We have

$$A(0)=\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \end{vmatrix} = E \in M_n(R)$$

and $A(x) = E + x \cdot I_n$, where

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in M_n(R)$$

$$A(x) \cdot A(y) = (E + x \cdot I_n)(E + y \cdot I_n) = E^2 + (x+y)E + xyI_n$$

and because $E^2 = nE$, yields that

$$A(x) \cdot A(y) = nE + (x+y)E + xyI_n = (x+y+n)E + xyI_n$$

So

$$A(0) \cdot A(1) = (n+1)E \text{ and } A(2) \cdot A(3) = (n+5)E + 6I_n$$

Hence

$$A(0) \cdot A(1) \cdot A(2) \cdot A(3) =$$

$$= (n+1)E((n+5)E + 6I_n) = (n+1)(n+5)E^2 + 6(n+1)E =$$

$$= n(n+1)(n+5)E + 6(n+1)E = (n+1)\left(n^2 + 11n + 6\right)E$$

W8. (Solution by the proposer.) For any two almost everywhere nonzero sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$ we will use notation $a_n \sim b_n$ and say that sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ asymptotically equal.

Let $H(n) := 1^1 2^2 ... n^n$ (hyperfactorial). Then
$sf(n) = 1^n 2^{n-1} ... n^1 = \frac{(n!)^{n+1}}{H(n)}$ and, since $\frac{n}{\sqrt[n]{n!}} \sim e$,
$\sqrt[n^2]{n!} \sim 1$, we have

$$\frac{\sqrt{n}}{\sqrt[n^2]{sf(n)}} = \frac{\sqrt[n^2]{H(n)}\sqrt{n}}{(n!)^{\frac{n+1}{n^2}}} = \frac{\sqrt[n^2]{H(n)}}{\sqrt{n}} \cdot \frac{n}{\sqrt[n]{n!}} \cdot \frac{1}{\sqrt[n^2]{n!}} \sim e \cdot \frac{\sqrt[n^2]{H(n)}}{\sqrt{n}}.$$

1. First we will find $\lim\limits_{n\to\infty} \frac{\sqrt{n}}{\sqrt[n^2]{sf(n)}}$.

We have

$$\ln \frac{\sqrt[n^2]{H(n)}}{\sqrt{n}} = \frac{1}{n^2}\sum_{k=1}^{n} k\ln k - \frac{1}{2}\ln n = \frac{1}{n^2}\sum_{k=1}^{n} k\ln k - \frac{1}{n^2}\cdot\frac{n(n+1)}{2}\ln n + \frac{\ln n}{2n} =$$

$$= \frac{1}{n^2}\sum_{k=1}^{n}(k\ln k - k\ln n) + \frac{\ln n}{2n} = \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\ln\frac{k}{n} + \frac{\ln n}{2n}.$$

Since $\lim\limits_{x\to+0}(x\ln x) = 0$ then function $f(x) := \{\ x\ln x,\ \text{if } x > 00,\ \text{if } x = 0$ is continuous on $[0,\infty)$ and, therefore,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\ln\frac{k}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx = \lim_{\varepsilon\to+0}\int_\varepsilon^1 x\ln x dx =$$

$$= \lim_{\varepsilon\to+0}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)_\varepsilon^1 = \lim_{\varepsilon\to+0}\left(-\frac{\varepsilon^2}{2}\ln\varepsilon - \frac{1}{4} + \frac{\varepsilon^2}{4}\right) = -\frac{1}{4}.$$

Hence,

$$\lim_{n\to\infty}\ln\frac{\sqrt[n^2]{H(n)}}{\sqrt{n}} = -\frac{1}{4} \iff \lim_{n\to\infty}\frac{\sqrt[n^2]{H(n)}}{\sqrt{n}} = e^{-\frac{1}{4}}$$

and, therefore,

$$\frac{\sqrt{n}}{\sqrt[n^2]{sf(n)}} \sim e\cdot\frac{\sqrt[n^2]{H(n)}}{\sqrt{n}} \sim e\cdot e^{-\frac{1}{4}} = e^{\frac{3}{4}} \iff \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt[n^2]{sf(n)}} = e^{\frac{3}{4}}.$$

2. Let $a_n := \dfrac{\sqrt{n}}{\sqrt[n^2]{sf(n)}}$.Since $a_n \sim e^{\frac{3}{4}}$ then $\dfrac{(n+1)\,a_{n+1}}{na_n} \sim \dfrac{a_{n+1}}{a_n} \sim 1$ and, therefore,

$$\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}} - \frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}} = (n+1)\,a_{n+1} - na_n = a_n \cdot n\left(\frac{(n+1)\,a_{n+1}}{na_n} - 1\right) \sim$$

$$\sim e^{\frac{3}{4}} \ln\left(\frac{(n+1)\,a_{n+1}}{na_n}\right)^n = e^{\frac{3}{4}}\left(n\ln\left(1+\frac{1}{n}\right) + \ln\left(\frac{a_{n+1}}{a_n}\right)^n\right).$$

Consider now

$$\left(\frac{a_{n+1}}{a_n}\right)^n = \frac{(n+1)^{\frac{n}{2}}}{\sqrt[n+1]{sf(n+1)}} \cdot \frac{\sqrt[n]{sf(n)}}{n^{\frac{n}{2}}} \cdot \frac{\sqrt[(n+1)^2]{sf(n+1)}}{\sqrt{n+1}} \cdot \sqrt{n+1} =$$

$$= \left(1+\frac{1}{n}\right)^{\frac{n}{2}} \cdot \frac{\sqrt[(n+1)^2]{sf(n+1)}}{\sqrt{n+1}} \cdot \frac{\sqrt[n]{sf(n)}}{\sqrt[n+1]{sf(n+1)}} \cdot \sqrt{n+1} \sim$$

$$\sim e^{\frac{1}{2}} \cdot e^{-\frac{3}{4}} \cdot \frac{\sqrt[n]{sf(n)}}{\sqrt[n+1]{sf(n+1)}} \cdot \sqrt{n+1} =$$

$$= e^{-\frac{1}{4}} \cdot \frac{\sqrt[n]{sf(n)}}{\sqrt[n+1]{sf(n+1)}} \cdot \sqrt{n+1}.$$

For further we need

Lemma.

$$\sqrt[n]{H(n)} \sim e^{-\frac{n}{4}} \cdot n^{\frac{n+1}{2}}$$

Proof. Note that

$$\sqrt[n]{H(n)} \sim e^{-\frac{n}{4}} \cdot n^{\frac{n+1}{2}} \iff e^{\frac{n}{4}} \cdot n^{-\frac{n+1}{2}} \cdot \left(1^1 \cdot 2^2 \cdot \ldots \cdot n^n\right)^{\frac{1}{n}} \sim 1 \iff$$

$$\iff \sqrt[n]{b_n} \sim 1,$$

where

$$c_n := e^{\frac{n^2}{4}} \cdot n^{-\frac{n(n+1)}{2}} \cdot 1^1 \cdot 2^2 \cdot \ldots \cdot n^n .$$

We have

$$\frac{c_{n+1}}{c_n}=\frac{e^{\frac{(n+1)^2}{4}}\cdot n^{\frac{n(n+1)}{2}}\cdot 1^1\cdot 2^2\cdot\ldots\cdot n^n\cdot(n+1)^{n+1}}{(n+1)^{\frac{(n+1)(n+2)}{2}}\cdot e^{\frac{n^2}{4}}\cdot 1^1\cdot 2^2\cdot\ldots\cdot n^n}=$$

$$=\frac{e^{\frac{2n+1}{4}}\cdot n^{\frac{n(n+1)}{2}}\cdot(n+1)^{n+1}}{(n+1)^{\frac{(n+1)(n+2)}{2}}}=\frac{e^{\frac{2n+1}{4}}\cdot n^{\frac{n(n+1)}{2}}}{(n+1)^{\frac{(n+1)(n+2)}{2}-(n+1)}}=$$

$$=\frac{e^{\frac{2n+1}{4}}\cdot n^{\frac{n(n+1)}{2}}}{(n+1)^{\frac{n(n+1)}{2}}}=\frac{e^{\frac{2n+1}{4}}}{\left(1+\frac{1}{n}\right)^{\frac{n(n+1)}{2}}}$$

and then

$$\ln\frac{c_{n+1}}{c_n}=\frac{2n+1}{4}-\frac{n(n+1)}{2}\ln\left(1+\frac{1}{n}\right).$$

Since $\ln(1+t)=t-\frac{t^2}{2}+o\left(t^2\right)$ then

$$\ln\frac{c_{n+1}}{c_n}=\frac{2n+1}{4}-\frac{n(n+1)}{2}\left(\frac{1}{n}-\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right)=$$

$$=\ln\frac{b_{n+1}}{b_n}=\frac{2n+1}{4}-\frac{n+1}{2}+\frac{n+1}{4n}+\frac{n(n+1)}{2}o\left(\frac{1}{n^2}\right)=$$

$$=\frac{1}{4n}+\frac{n(n+1)}{2}o\left(\frac{1}{n^2}\right)$$

and, therefore,

$$\lim_{n\to\infty}\ln\frac{c_{n+1}}{c_n}=\lim_{n\to\infty}\left(\frac{1}{4n}+\frac{n(n+1)}{2}o\left(\frac{1}{n^2}\right)\right)=0.$$

Hence, $\lim\limits_{n\to\infty}\frac{c_{n+1}}{c_n}=e^0=1$ and, by Multiplicative Stolz Theorem, we have

$$\lim_{n\to\infty}\sqrt[n]{c_n}=1.$$

Since

$$\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}=\frac{\sqrt[n]{n!}}{n}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\cdot\frac{n}{n+1}\sim e^{-1}\cdot e\cdot 1=1$$

then, applying Lemma, we obtain

$$\frac{\sqrt[n]{sf(n)}}{\sqrt[n+1]{sf(n+1)}} = \frac{(n!)^{\frac{n+1}{n}}}{\sqrt[n]{H(n)}} \cdot \frac{\sqrt[n+1]{H(n+1)}}{((n+1)!)^{\frac{n+1}{n}}} =$$

$$= \frac{1}{n+1} \cdot \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \cdot \frac{\sqrt[n+1]{H(n+1)}}{\sqrt[n]{H(n)}} \sim$$

$$\frac{1}{n+1} \cdot \frac{\sqrt[n+1]{H(n+1)}}{\sqrt[n]{H(n)}} \sim \frac{1}{n+1} \cdot \frac{e^{-\frac{n+1}{4}}(n+1)^{\frac{n+2}{2}}}{e^{-\frac{n}{4}} n^{\frac{n+1}{2}}} =$$

$$= \frac{e^{-\frac{1}{4}}}{\sqrt{n+1}} \cdot \left(1+\frac{1}{n}\right)^{\frac{n+1}{2}} \sim \frac{e^{-\frac{1}{4}}}{\sqrt{n+1}} \cdot e^{\frac{1}{2}} \sim \frac{e^{\frac{1}{4}}}{\sqrt{n+1}}.$$

Hence,

$$\left(\frac{a_{n+1}}{a_n}\right)^n \sim e^{-\frac{1}{4}} \cdot \frac{\sqrt[n]{sf(n)}}{\sqrt[n+1]{sf(n+1)}} \cdot \sqrt{n+1} \sim e^{-\frac{1}{4}} \cdot \frac{e^{\frac{1}{4}}}{\sqrt{n+1}} \cdot \sqrt{n+1} = 1$$

and, therefore,

$$\lim_{n\to\infty}\left(\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}} - \frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}\right) =$$

$$= e^{\frac{3}{4}} \lim_{n\to\infty}\left(n\ln\left(1+\frac{1}{n}\right) + \ln\left(\frac{a_{n+1}}{a_n}\right)^n\right) =$$

$$= e^{\frac{3}{4}}(1+\ln 1) = e^{\frac{3}{4}}.$$

Remark. *Solution can be shorter in case of using the asymptotic* $\sqrt[n]{H(n)} \sim e^{-\frac{n}{4}} n^{\frac{n+1}{2}}$ *(see [1] or [2]).*

1. Solutions in real analysis, Masayoshi Hata, page 2,Problem1.3
2. Problems and theorems in Analysis I, G.Pólya, G.Szegő, page 50, Problem 15.

Second solution. We will make use several times of the Cesaro–Stolz theorem in both the most known versions (C–S1 and C–S2)

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}, \quad \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}$$

then, applying Lemma, we obtain

$$\frac{\sqrt[n]{sf(n)}}{\sqrt[n+1]{sf(n+1)}} = \frac{(n!)^{\frac{n+1}{n}}}{\sqrt[n]{H(n)}} \cdot \frac{\sqrt[n+1]{H(n+1)}}{((n+1)!)^{\frac{n+1}{n}}} =$$

$$= \frac{1}{n+1} \cdot \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \cdot \frac{\sqrt[n+1]{H(n+1)}}{\sqrt[n]{H(n)}} \sim$$

$$\frac{1}{n+1} \cdot \frac{\sqrt[n+1]{H(n+1)}}{\sqrt[n]{H(n)}} \sim \frac{1}{n+1} \cdot \frac{e^{-\frac{n+1}{4}}(n+1)^{\frac{n+2}{2}}}{e^{-\frac{n}{4}}n^{\frac{n+1}{2}}} =$$

$$= \frac{e^{-\frac{1}{4}}}{\sqrt{n+1}} \cdot \left(1+\frac{1}{n}\right)^{\frac{n+1}{2}} \sim \frac{e^{-\frac{1}{4}}}{\sqrt{n+1}} \cdot e^{\frac{1}{2}} \sim \frac{e^{\frac{1}{4}}}{\sqrt{n+1}}.$$

Hence,

$$\left(\frac{a_{n+1}}{a_n}\right)^n \sim e^{-\frac{1}{4}} \cdot \frac{\sqrt[n]{sf(n)}}{\sqrt[n+1]{sf(n+1)}} \cdot \sqrt{n+1} \sim e^{-\frac{1}{4}} \cdot \frac{e^{\frac{1}{4}}}{\sqrt{n+1}} \cdot \sqrt{n+1} = 1$$

and, therefore,

$$\lim_{n\to\infty} \left(\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}} - \frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}} \right) =$$

$$= e^{\frac{3}{4}} \lim_{n\to\infty} \left(n \ln\left(1+\frac{1}{n}\right) + \ln\left(\frac{a_{n+1}}{a_n}\right)^n \right) =$$

$$= e^{\frac{3}{4}}(1+\ln 1) = e^{\frac{3}{4}}.$$

Remark. *Solution can be shorter in case of using the asymptotic* $\sqrt[n]{H(n)} \sim e^{-\frac{n}{4}} n^{\frac{n+1}{2}}$ *(see [1] or [2]).*

1. Solutions in real analysis, Masayoshi Hata, page 2,Problem1.3
2. Problems and theorems in Analysis I, G.Pólya, G.Szegő, page 50, Problem 15.

Second solution. We will make use several times of the Cesaro–Stolz theorem in both the most known versions (C–S1 and C–S2)

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}, \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$$

provided that $b_n \to \infty$

$$\left(\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}} - \frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}\right) = \frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}\left(\frac{\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}}}{\frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}} - 1\right) =$$

$$= \frac{\sqrt{n}}{\sqrt[n^2]{sf(n)}} \frac{\left(\frac{\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}}}{\frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}} - 1\right)}{\ln\left(\frac{\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}}}{\frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}}\right)} \ln\left(\frac{\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}}}{\frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}}\right)^n$$

We prove the

Lemma.

$$\lim_{n\to\infty} \sqrt{n}/\sqrt[n^2]{sf(n)} = e^{3/4}$$

Proof of the Lemma The first step is $\lim_{n\to\infty} \sqrt[n^2]{sf(n)} = \infty$. We do the logarithm and study $\lim_{n\to\infty} \frac{1}{n^1}\ln(sf(n))$. For doing this we use C–S2 that is

$$\lim_{n\to\infty} \frac{\ln\frac{sf(n+1)}{sf(n)}}{(n+1)^2 - n^2} = \lim_{n\to\infty} \frac{\ln(n+1)!}{2n+1}$$

Let's apply aggain CS–2 and get

$$\lim_{n\to\infty} \frac{(n+2)!}{(n+1)!} 2n+3-(2n+1) = \lim_{n\to\infty} \frac{n+1}{2} = \infty$$

whence $\sqrt[n^2]{sf(n)} \to \infty$. This allows us to use

$$\frac{\sqrt{n}}{\sqrt[n^2]{sf(n)}} = \left(\frac{n^{\frac{n^2}{2}}}{sf(n)}\right)^{\frac{1}{n^2}}$$

The logarithm yields

$$\frac{1}{n^2}\ln\frac{n^{\frac{n^2}{2}}}{sf(n)}$$

C–S2 gives

$$\lim_{n\to\infty}\frac{1}{2n+1}\ln\frac{(n+1)^{\frac{(n+1)^2}{2}}}{sf(n+1)}\frac{sf(n)}{n^{\frac{n^2}{2}}}=\lim_{n\to\infty}\frac{1}{2n+1}\ln\frac{(n+1)^{\frac{(n+1)^2}{2}}}{(n+1)!n^{\frac{n^2}{2}}}$$

A further application of C–S2 gives

$$\lim_{n\to\infty}\frac{1}{2n+3-(2n+1)}\ln\frac{(n+2)^{\frac{(n+2)^2}{2}}}{(n+2)!(n+1)^{\frac{(n+1)^2}{2}}}\frac{(n+1)!n^{\frac{n^2}{2}}}{(n+1)^{\frac{(n+1)^2}{2}}}=$$

$$=\lim_{n\to\infty}\frac{1}{2}\ln\frac{(n+2)^{\frac{(n+2)^2}{2}-1}n^{\frac{n^2}{2}}}{(n+1)^{(n+1)^2}}$$

Performing the asymptotic expansion of the various powers, we come to

$$\lim_{n\to\infty}\frac{1}{2}\ln e^{\frac{3}{2}-\frac{1}{n}+\frac{17}{12n^2}+O(n^{-1})}=\frac{3}{4}$$

and exponentiating $e^{3/4}$. *End of the proof of the Lemma*

It follows

$$\lim_{n\to\infty}\frac{\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}}}{\frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}}=\lim_{n\to\infty}\frac{n+1}{n}\frac{\frac{\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}}}{\frac{\sqrt{n}}{\sqrt[n^2]{sf(n)}}}=1\cdot\frac{e^{\frac{3}{4}}}{e^{\frac{3}{4}}}=1$$

Now we prove that

$$\lim_{n\to\infty}\left(\frac{\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}}}{\frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}}\right)^n=\lim_{n\to\infty}\left(\frac{(n+1)^{\frac{3}{2}}}{n^{\frac{3}{2}}}\right)^n\left(\frac{\sqrt[n^2]{sf(n)}}{\sqrt[(n+1)^2]{sf(n+1)}}\right)^n=$$

$$=\lim_{n\to\infty}e^{\frac{3}{2}}\left(\frac{\sqrt[n^2]{sf(n)}}{\sqrt[(n+1)^2]{sf(n+1)}}\right)^n=e$$

To this end first we prove.

Lemma 1.

$$\lim_{n\to\infty}\sqrt[n^3]{sf(n)}=1.$$

Proof of Lemma1

$$(sf(n))^{\frac{1}{n^3}} = \left(\frac{(sf(n))^{\frac{1}{n^2}}}{\sqrt{n}}\right)^{\frac{1}{n}} (\sqrt{n})^{\frac{1}{n}}$$

whose limit is clearly one because both the factors tend to one. *End of proof of Lemma 1*

Now we come back to

$$\left(\frac{\sqrt[n^2]{sf(n)}}{\sqrt[(n+1)^2]{sf(n+1)}}\right)^n = \left(\frac{sf(n)}{(sf(n+1))^{\frac{n^2}{(n+1)^2}}}\right)^{\frac{1}{n}} =$$

$$= \left(\frac{sf(n)}{(sf(n+1))}(sf(n+1))^{1-\frac{n^2}{(n+1)^2}}\right)^{\frac{1}{n}}$$

Now we use C–S1 and get

$$\lim_{n\to\infty} \frac{sf(n+1)}{sf(n+2)} \frac{sf(n+1)}{sf(n)} \frac{(sf(n+2))^{\frac{2n+3}{(n+2)^2}}}{(sf(n+1))^{\frac{2n+1}{(n+1)^2}}} =$$
$$= \lim_{n\to\infty} \frac{1}{n+2}((n+2)!)^{\frac{2n+3}{(n+2)^2}} (sf(n+1))^{\frac{2n+3}{(n+2)^2}-\frac{2n+1}{(n+1)^2}}$$

$$\frac{2n+3}{(n+2)^2} = \frac{2}{n} + O(\frac{1}{n^2}), \qquad \frac{2n+3}{(n+2)^2} - \frac{2n+1}{(n+1)^2} = -\frac{2}{n^2} + O(\frac{1}{n^3})$$

Then we write using Stirling's formula

$$\lim_{n\to\infty}\frac{1}{n+2}((n+2)!)^{\frac{2n+3}{(n+2)^2}}(sf(n+1))^{\frac{2n+3}{(n+2)^2}-\frac{2n+1}{(n+1)^2}}=$$

$$\lim_{n\to\infty}\frac{1}{n+2}e^{((n+2)\ln(n+2)-n-2)\left(\frac{2}{n}+O(\frac{1}{n^2})\right)}\left(\sqrt{2\pi(n+2)}(1+o(1))\right)^{\frac{2}{n}+O(\frac{1}{n^2})}\cdot$$

$$\cdot\frac{1}{n}\frac{n}{(sf(n+1))^{\frac{2}{n^2}}\left(1+O(\frac{1}{n})\right)}=$$

$$=\lim_{n\to\infty}\frac{1}{n(n+2)}e^{((n+2)\ln(n+2)-n-2)\frac{2}{n}}\frac{n}{(sf(n+1))^{\frac{2}{n^2}}}=$$

$$=\lim_{n\to\infty}\frac{1}{n(n+2)}e^{2\ln n-2+O(\frac{\ln n}{n})}e^{\frac{3}{2}}=e^{-2+\frac{3}{2}}$$

Finally

$$\lim_{n\to\infty}e^{\frac{3}{2}}\left(\frac{\sqrt[n^2]{sf(n)}}{\sqrt[(n+1)^2]{sf(n+1)}}\right)^n=\lim_{n\to\infty}e^{\frac{3}{2}}e^{-2+\frac{3}{2}}=e$$

and then

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt[n^2]{sf(n)}}\frac{\left(\frac{\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}}}{\frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}}-1\right)}{\ln\left(\frac{\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}}}{\frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}}\right)}\ln\left(\frac{\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}}}{\frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}}}\right)^n=e^{\frac{3}{4}}\cdot 1\cdot 1=e^{\frac{3}{4}}$$

and this concludes the proof.

Paolo Perfetti

W9. (Solution by the proposer.) Consider recurrence

$$x_{n+1}=\frac{1}{k}\left(x_n+x_{n-1}+x_{n-2}+x_{n-k+1}\right),n\geq k-1$$

with $x_n\in\mathbb{C},n\in\mathbb{N}\cup\{0\}$.

Easy to see that the sequence $(x_n)_{n\geq 0}$, originally defined by linear homogeneous recurrence of k-degree can be equivalently defined by the following linear nonhomogeneous recurrence of degree $k-1$