Superstable triangles.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Let $\Delta(x,y,z) = 2xy + 2yz + 2zx - x^2 - y^2 - z^2$. Find all triangles with sidelengths a,b,c such that $\Delta(a^n,b^n,c^n) > 0$ for any $n \in \mathbb{N}$.

Solution.

Note that $\Delta(x^2,y^2,z^2) = (x+y+z)(x+y-z)(x-y+z)(-x+y+z)$ and for positive x,y,z we have equivalency $\Delta(x^2,y^2,z^2) > 0 \Leftrightarrow \begin{cases} x+y > z \\ y+z > x \\ z+x > y \end{cases}$.

Due symmetry and homogeneity of $\Delta(a^n,b^n,c^n) > 0$ WLOG we assume that $a \geq b \geq 1$. Then for any $n \in \mathbb{N}$ we have

$$\begin{cases} \Delta(a^{2n},b^{2n},c^{2n}) > 0 \\ a \geq b \geq c = 1 \end{cases} \Leftrightarrow \begin{cases} b^n + 1 > a^n \\ a \geq b \geq c = 1 \end{cases}.$$

Suppose that $a > b$, then $a^n = (b + (a-b))^n > b^n + n(a-b)b^{n-1} > b^n + n(a-b) > b^n + 1$ for any $n > \dfrac{1}{a-b}$. It is contradict to $b^n + 1 > a^n$ which holds for any $n \in \mathbb{N}$.

Thus $a = b$ and, therefore, triangle should be isosceles with two equal sides, which not less then third one.

Let now $a = b \geq c$ then $\Delta(a^n,b^n,c^n) = 2a^nb^n + 2b^nc^n + 2c^na^n - a^{2n} - b^{2n} - c^{2n} = 4c^na^n - c^{2n} \geq 3c^{2n} > 0.$■