W7. Find the differential equation of the fourth order with constant coefficients, having as particular solutions e^{-x}, e^{x}, double e^{2x}, and its non-homogeneous term being $2x^2$. Solve after, its Cauchy problem: $y(0) = 3, y'(0) = y''(0) = 2, y'''(0) = 1$.

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Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Characteristic equation for the differential equation is $(k^2-1)(k-2)^2 = k^4 - 4k^3 + 3k^2 + 4k - 4$ and, therefore, it is $y'''' - 4y''' + 3k'' + 4y' - 4y = 2x^2$.
By substitution partial solution in the form $u = ax^2 + bx + c$ in this equation we obtain $3(2a) + 4(2ax+b) - 4(ax^2+bx+c) = 2x^2 \Leftrightarrow$
$-4ax^2 + 4(2a-b)x + 2(3a+2b-2c) = 2x^2$. Hence,

$$\begin{cases} -4a = 2 \\ 2a - b = 0 \\ 3a + 2b - 2c = 0 \end{cases}, \Leftrightarrow \left[a = -\frac{1}{2}, b = -1, c = -\frac{7}{4}\right]$$

Hence, the general solution of the equation is sum of general solution of homogeneous equation $y'''' - 4y''' + 3k'' + 4y' - 4y = 0$ and obtained above partial solution, that is $y(x) = c_1e^x + c_2e^{-x} + (c_3x + c_4)e^{2x} - \frac{x^2}{2} - x - \frac{7}{4}$.
Since $y'(x) = -x + e^xc_1 - c_2e^{-x} + 2e^{2x}(c_4 + xc_3) + e^{2x}c_3 - 1$,
$y''(x) = e^xc_1 + c_2e^{-x} + 2c_3e^{2x} + 4e^{2x}(c_4 + xc_3) + 2e^{2x}c_3 - 1$,
$y'''(x) = e^xc_1 - c_2e^{-x} + 8c_3e^{2x} + 8e^{2x}(c_4 + xc_3) + 4e^{2x}c_3$
Using $y(0) = 3, y'(0) = y''(0) = 2, y'''(0) = 1$ we obtain

$$\begin{cases} c_1 + c_2 + c_4 - \frac{7}{4} = 3 \\ c_1 - c_2 + 2c_4 + c_3 - 1 = 2 \\ c_1 + c_2 + 2c_3 + 4c_4 + 2c_3 - 1 = 2 \\ c_1 - c_2 + 8c_3 + 8c_4 + 4c_3 = 1 \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 + c_4 = \frac{19}{4} \\ c_1 - c_2 + 2c_4 + c_3 = 3 \\ c_1 + c_2 + 2c_3 + 4c_4 + 2c_3 = 3 \\ c_1 - c_2 + 8c_3 + 8c_4 + 4c_3 = 1 \end{cases} \Leftrightarrow$$

$$\left[c_1 = \frac{11}{2}, c_2 = \frac{1}{2}, c_3 = \frac{1}{2}, c_4 = -\frac{5}{4}\right].$$

Thus, $y(x) = \frac{11}{2}e^x + \frac{1}{2}e^{-x} + \left(\frac{1}{2}x - \frac{5}{4}\right)e^{2x} - \frac{x^2}{2} - x - \frac{7}{4}$.