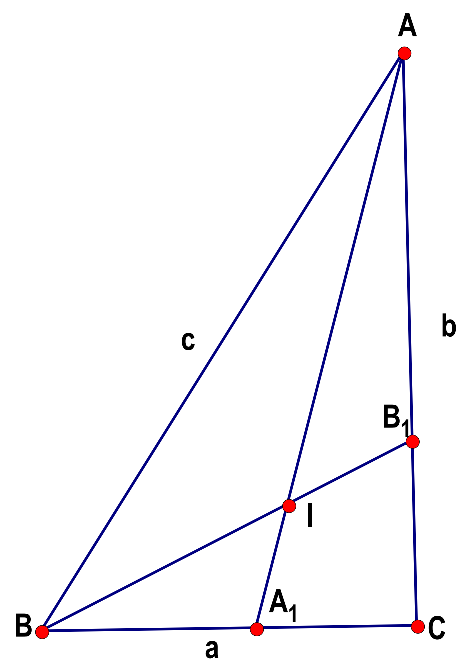
Maximum area of bisectorial quadrilateral in right triangle.
Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

Let $\triangle ABC$ be a right triangle with right angle in C and let be intersection point of bisectors AA_1, BB_1 of acute angles $\angle A$ and $\angle B$, respectively.
Find the right triangle with greatest value of ratio of the "bisectoria" quadrilateral A_1CB_1I area to the triangle $\triangle ABC$ area.



Solution.

Let $F := [ABC]$. Since $[A_1CB_1I] = [A_1AB] - [IAB_1]$ and

$\dfrac{CA_1}{BC} = \dfrac{b}{b+c}, \dfrac{AB_1}{AC} = \dfrac{c}{a+c}, \dfrac{IB_1}{BB_1} =$

$\dfrac{b}{a+b+c}$ then $[A_1AC] = \dfrac{bF}{b+c}, [ABB_1] = \dfrac{cF}{a+c}, [IAB_1] = \dfrac{b[ABB_1]}{a+b+c} = \dfrac{bcF}{(a+b+c)(a+c)}$,

$$[A_1CB_1I] = \frac{bF}{b+c} - \frac{bcF}{(a+b+c)(a+c)} = \frac{bF\left((a+c)^2+ab+bc-bc-c^2\right)}{(a+b+c)(a+c)(b+c)} =$$

$$\frac{Fab(2c+a+b)}{(a+b+c)(a+c)(b+c)} = \frac{F\left((a+b)^2-c^2\right)(2c+a+b)}{2(a+b+c)(a+c)(b+c)} = \frac{F(a+b-c)(2c+a+b)}{2(a+c)(b+c)}.$$

Thus $\dfrac{[A_1CB_1I]}{F} = \dfrac{(a+b-c)(2c+a+b)}{2(a+c)(b+c)}$. Now we will find $\max \dfrac{(a+b-c)(2c+a+b)}{2(a+c)(b+c)}$.

Due to homogeneity of $\dfrac{(a+b-c)(2c+a+b)}{(a+c)(b+c)}$ we can assume that $c=1$.

Since $a^2+b^2=1$ then, denoting $t := a+b$ obtain that $t \le \sqrt{2}$ ($\Leftrightarrow a+b \le \sqrt{2(a^2+b^2)}$),

$(a+1)(b+1) = 1+t+\dfrac{t^2-1}{2} = \dfrac{(t+1)^2}{2}, \dfrac{(a+b-c)(2c+a+b)}{2(a+c)(b+c)} = \dfrac{(t-1)(2+t)}{2(a+1)(b+1)} =$

$\dfrac{(t-1)(2+t)}{(t+1)^2}$ and, therefore, $\max\left(\dfrac{(a+b-c)(2c+a+b)}{2(a+c)(b+c)}\right) = \max\limits_{0<t\le\sqrt{2}}\left(\dfrac{(t-1)(t+2)}{(t+1)^2}\right)$.

Since $t+1 \le \sqrt{2}+1 \iff \dfrac{1}{t+1} \ge \sqrt{2}-1$ then $\dfrac{(t-1)(t+2)}{(t+1)^2} = \dfrac{t^2+t-2}{(t+1)^2} = 1 - \dfrac{1}{t+1} - \dfrac{2}{(t+1)^2} \le 1 - \left(\sqrt{2}-1\right) - 2\left(\sqrt{2}-1\right)^2 = 2-\sqrt{2}-6+4\sqrt{2} = 3\sqrt{2}-4$ and equaliy occurs iff

$t = \sqrt{2}$. Thus, $\max \dfrac{[A_1CB_1I]}{F} = 3\sqrt{2}-4$ and can be attained only iff

$a = b = \dfrac{\sqrt{2}\,c}{2}$ because

$$\begin{cases} a^2+b^2 = c^2 \\ a+b = \sqrt{2}\,c \end{cases} \iff \begin{cases} a^2+b^2 = c^2 \\ a = b \end{cases}.$$