W60,2018-Infinite sum for a sequence defined recursively.

Problem with a solution proposed by Arkady Alt, San Jose , California, USA.

Let x, a, h be arbitrary real numbers such that $x > 0, a \geq -1$, $h > 0$ and let sequence (x_n) defined recursively by $x_1 = x,\ x_{n+1} = \dfrac{n+a}{n+a+h} x_n,\ n \in N \cup \{0\}$.

Explore for which h the infinite sum $\sum_{n=1}^{\infty} x_n$ converges and find it in the case of convergence.

Solution.

If $0 < h \leq 1$ then from $(n+a+h)x_{n+1} = (1-h)\sum_{k=1}^{n} x_k + (a+h)x \geq (a+h)x$ follows

$x_{n+1} \geq \dfrac{(a+h)x}{n+a+h}$.

Since $x_n \geq \dfrac{(a+h)x}{n-1+a+h}, n \in N$ and $\sum_{n=1}^{\infty} \dfrac{(a+h)x}{n-1+a+h}$ diverge, then $\sum_{n=1}^{\infty} x_n$ diverge too.

To consider case $h > 1$ we need the lemma

Lemma.

For any positive t and $a \geq -1$, $h > 1$ holds inequality

(1) $\left(\dfrac{t+a}{t+1+a}\right)^h < \dfrac{t+a}{t+a+h} < \left(\dfrac{t-1+a+h}{t+a+h}\right)^h.$

Proof.

By Bernoulli Inequality $(1+u)^\alpha \geq 1+\alpha u$, $\alpha \geq 1, u > -1$ we obtain

$$\left(\frac{t-1+a+h}{t+a+h}\right)^h = \left(1 - \frac{1}{t+a+h}\right)^h > 1 - \frac{h}{t+a+h} = \frac{t+a}{t+a+h}$$

and

$$\left(\frac{t+a+1}{t+a}\right)^h = \left(1 + \frac{1}{t+a}\right)^h > 1 + \frac{h}{t+a} = \frac{t+a+h}{t+a} \Leftrightarrow$$

$$\frac{t+a}{t+a+h} > \left(\frac{t+a}{t+a+1}\right)^h. \blacksquare$$

Since $x_{n+1} = \dfrac{n+a}{n+a+h} x_n \Leftrightarrow (n+a+h)x_{n+1} = (n+a+h-1)x_n - (h-1)x_n \Leftrightarrow$

$(h-1)x_n = (n+a+h-1)x_n - (n+a+h)x_{n+1},\ n \in \mathbb{N}$

then

$$(h-1)\sum_{k=1}^{n} x_k = \sum_{k=1}^{n}((k+a+h-1)x_k - (k+a+h)x_{k+1}) =$$

$$(a+h)x_1 - (n+a+h)x_{n+1} = (a+h)x - (n+a+h)x_{n+1}$$

$$\sum_{k=1}^{n} x_k = \frac{(a+h)x - (n+a+h)x_{n+1}}{h-1}.$$

From the other hand

$$\prod_{k=1}^{n} x_{k+1} = \prod_{k=1}^{n} \frac{x_k(k+a)}{k+a+h} \Leftrightarrow x_{n+1} \cdot \prod_{k=2}^{n} x_k = x_1 \prod_{k=2}^{n} x_k \cdot \prod_{k=1}^{n} \frac{k+a}{k+a+h} \Leftrightarrow x_{n+1} = x \prod_{k=1}^{n} \frac{k+a}{k+a+h}.$$

Then, applying inequality (**1**), we obtain

$$x\prod_{k=1}^{n}\left(\frac{k+a}{k+1+a}\right)^h < x_{n+1} < x\prod_{k=1}^{n}\left(\frac{k-1+a+h}{k+a+h}\right)^h.$$

Since

$\prod_{k=1}^{n} \frac{k+a}{k+1+a} = \frac{a+1}{n+1+a}$, $\prod_{k=1}^{n} \frac{k-1+a+h}{k+a+h} = \frac{a+h}{n+a+h}$ and $\frac{a+1}{n+1+a} > \frac{a+1}{n+a+h}$ then

$$x\left(\frac{a+1}{n+1+a}\right)^h < x_{n+1} < x\left(\frac{a+h}{n+a+h}\right)^h \Rightarrow$$

(**2**) $$x\left(\frac{a+1}{n+a+h}\right)^h < x_{n+1} < x\left(\frac{a+h}{n+a+h}\right)^h.$$

Multiplying (**2**) by $n+a+h$ we obtain inequality

(**3**) $$\frac{x(a+1)^h}{(n+1+a)^{h-1}} < (n+a+h)x_{n+1} < \frac{x(a+h)^h}{(n+1+a)^{h-1}}.$$

Since $\lim_{n\to\infty} \frac{x(a+1)^h}{(n+1+a)^{h-1}} = \lim_{n\to\infty} \frac{x(a+h)^h}{(n+1+a)^{h-1}} = 0,$ then

$$\lim_{n\to\infty}(n+a+h)x_{n+1} = 0$$ and

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k = \lim_{n\to\infty} \frac{(a+h)x-(n+a+h)x_{n+1}}{h-1} = \frac{(a+h)x}{h-1}.$$

So, the infinite sum $\sum_{n=1}^{\infty} x_n$ diverge if $0 < h \le 1$ and converge to $\frac{(a+h)x}{h-1}$ if $h > 1$.