$$f(x) = g(x) + g'(x) + g''(x)$$

By Rolle's theorem there is a value $c \in (a, b)$ such that

$$1 = \frac{1}{b-a} \ln\left(\frac{f(b)}{f(a)}\right) = \frac{\ln(f(b)) - \ln(f(a))}{b-a} = \frac{d}{dx} \ln(f(x))\,|_{x=c} = \frac{f'(c)}{f(c)}$$

So $0 = f'(c) - f(c) = g'''(c) - g(c)$ which is equivalent to the claimed equality.

Albert Stadler

Fourth solution.

$$\left(\frac{f(b)}{f(a)}\right) = b - a \iff \frac{\ln f(b) - \ln f(a)}{b-a} = 1$$

and the Lagrange's theorem or the mean–value–theorem yields the existence of a point $c \in (a, b)$ such that

$$(\ln(f(x))'\big|_{x=c} = 1 \iff \frac{f'(c)}{f(c)} = 1 \iff f'(c) = f(c)$$

$$f'(c) = \frac{1 + \frac{c}{\sqrt{1+c^2}}}{c + \sqrt{1+c^2}} - \frac{c}{(1+c^2)^{\frac{3}{2}}} - \frac{1}{(1+c^2)^{\frac{3}{2}}} + \frac{3c^2}{(1+c^2)^{\frac{5}{2}}} =$$
$$= \frac{1}{\sqrt{1+c^2}} - \frac{c+1}{(1+c^2)^{\frac{3}{2}}} + \frac{3c^2}{(1+c^2)^{\frac{5}{2}}} =$$

The equation $f'(c) = f(c)$ yields

$$\ln(c + \sqrt{1+c^2}) = -\frac{1}{(1+c^2)^{\frac{3}{2}}} + \frac{3c^2}{(1+c^2)^{\frac{5}{2}}}$$

that is

$$2c^2 = 1 + (1+c^2)^{5/2} \ln(c + \sqrt{1+c^2})$$

Paolo Perfetti

W6. Solution by the proposer. Let $S(\mathbf{x}_{\mathbb{N}}) = \sum\limits_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}}$ if series converges and $S_f(x_{\mathbb{N}}) = \infty$ if it diverges.
Let $\widetilde{D}_1 = \{x_{\mathbb{N}} \mid x_{\mathbb{N}} \in D_1 \text{ and } S(x_{\mathbb{N}}) \neq \infty\}$ .Since $\widetilde{D}_1$ isn't empty (because for for instance if $x_n = q^{n-1}, n \in \mathbb{N}$, where $q \in (0,1)$, we have

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} =$$

$$= \sum_{n=1}^{\infty} \frac{q^{3(n-1)}}{q^{n-1}+4q^n} = \sum_{n=1}^{\infty} \frac{q^{2(n-1)}}{1+4q} = \frac{1}{(1+4q)(1-q^2)}$$

then $\inf \{S(\mathbf{x}_{\mathbb{N}}) \mid \mathbf{x}_{\mathbb{N}} \in D_1\} = \inf \left\{S(\mathbf{x}_{\mathbb{N}}) \mid \mathbf{x}_{\mathbb{N}} \in \widetilde{D}_1\right\}$.

Let $S := \inf \left\{S(\mathbf{x}_{\mathbb{N}}) \mid \mathbf{x}_{\mathbb{N}} \in \widetilde{D}_1\right\}$. For any $\mathbf{x}_{\mathbb{N}} \in \widetilde{D}_1$ we have

$$S(\mathbf{x}_{\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} =$$

$$= \frac{1}{1+4x_2} + \sum_{n=2}^{\infty} \frac{x_n^3}{x_n+4x_{n+1}} = \frac{1}{1+4x_2} + x_2^2 \sum_{n=1}^{\infty} \frac{y_n^3}{y_n+4y_{n+1}} = \frac{1}{1+4x_2} + x_2^2 S(\mathbf{y}_{\mathbb{N}}),$$

where $y_n := \dfrac{x_{n+1}}{x_2}, n \in \mathbb{N}$.

Since $\mathbf{y}_{\mathbb{N}} \in \widetilde{D}_1$ ($1 = y_1 > y_2 > .... > y_n > ...$ and $S(\mathbf{y}_{\mathbb{N}}) = \dfrac{S(\mathbf{x}_{\mathbb{N}})}{x_2^2} - \dfrac{1}{1+4x_2}$) then $S(\mathbf{y}_{\mathbb{N}}) \geq S$ and, therefore, $S(\mathbf{x}_{\mathbb{N}}) \geq \dfrac{1}{1+4x_2} + x_2^2 S \implies S \geq \dfrac{1}{1+4x_2} + x_2^2 S \iff$ $S \geq \dfrac{1}{(1+4x_2)(1-x_2^2)}$.

We will find $\mu := \max\limits_{x\in(0,1)} h(x)$, where

$$h(x) := (1+4x)\left(1-x^2\right) = -4x^3 - x^2 + 4x + 1$$

Since $h'(x) = -12x^2 - 2x + 4 = -2(3x+2)(2x-1)$ then $\mu = \max\limits_{x\in(0,1)} h(x) = h\left(\dfrac{1}{2}\right) = \dfrac{9}{4}$ and, therefore, $S(\mathbf{x}_{\mathbb{N}}) \geq \dfrac{1}{\mu} = \dfrac{4}{9}$.

Since $S(\mathbf{x}_{\mathbb{N}}) = \dfrac{1}{(1+4q)(1-q^2)}$ for $x_n = q^{n-1}, n \in \mathbb{N}, q \in (0,1)$, then for $q = \dfrac{1}{2}$ we obtain

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n+4x_{n+1}} = \frac{1}{\left(1+4\cdot\frac{1}{2}\right)\left(1-\left(\frac{1}{2}\right)^2\right)} = \frac{4}{9}.$$

Second solution. If $x_k = 2^{-k+1}$ the equality occurs. Indeed

$$\sum_{n=1}^{\infty} \frac{2^{-3n+3}}{2^{-n+1}+4\cdot 2^{-n}} = \frac{4}{3}\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{4}{3}\frac{1}{4}\frac{1}{1-\frac{1}{4}} = \frac{4}{9}$$

Now consider the sequence $y_1 = 1$, $y_2 = \frac{1}{2} + \delta$, $0 < \delta < 1/2$, $y_k = 2^{-k+1}$ for $k \geq 3$.

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} - \sum_{n=1}^{\infty} \frac{y_n^3}{y_n + 4y_{n+1}} = \frac{4}{9} - \sum_{n=1}^{\infty} \frac{y_n^3}{y_n + 4y_{n+1}} =$$
$$= \frac{x_1^3}{x_1 + 4x_2} + \frac{x_2^3}{x_2 + 4x_3} - \frac{y_1^3}{y_1 + 4y_2} - \frac{y_2^3}{y_2 + 4y_3} =$$
$$= \frac{1}{1 + 4\frac{1}{2}} + \frac{\frac{1}{8}}{\frac{1}{2} + 4\frac{1}{4}} - \frac{1}{1 + 4(\frac{1}{2} + \delta)} - \frac{(\frac{1}{2} + \delta)^3}{\frac{1}{2} + \delta + 4\frac{1}{4}} =$$
$$= \frac{5}{12} - \frac{1}{3}\frac{1}{1 + \frac{4}{3}\delta} - \left(\frac{1}{2} + \delta\right)^3 \frac{2}{3}\frac{1}{1 + \frac{2}{3}\delta} =$$
$$= \frac{5}{12} - \frac{1}{3}\left(1 - \frac{4}{3}\delta\right) - \frac{1}{8}\frac{2}{3}\left(1 - \frac{2}{3}\delta\right) + O(\delta^2) = \frac{1}{2}\delta + O(\delta^2) > 0$$

Contradicting the statement.

Paolo Perfetti

Third solution. $Qx_1 = 1$, and sequence $\{x_n\}$ is the increasing geometric progression,

$\therefore x_n = 1 \times q^{n-1}$ $(0 < q < 1)$

Then $y_n = \frac{x_n^3}{x_n + 4x_{n+1}} = \frac{q^{3n-3}}{q^{n-1} + 4q^n} = \frac{q^{2n-2}}{1+4q}$

So

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} = \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{q^{2n-2}}{1 + 4q} = \sum_{n=1}^{\infty} \frac{1}{1 + 4q} \times \frac{1 - q^{2n}}{1 - q^2}$$

$Q0 < q < 1$, $n \to \infty$, $\therefore 1 - q^{2n} \to 1$,

$\therefore \sum\limits_{n=1}^{\infty} \frac{x^3}{x_n + 4x_{n+1}} = \sum\limits_{n=1}^{\infty} \frac{1}{1+4q} \times \frac{1}{1-q^2}$

$f(q) = (1 + 4q)\left(1 - q^2\right) = 1 - q^2 + 4q - 4q^3$ $(0 < q < 1)$

$f'(q) = -2q + 4 - 12q^2$

Let $f'(q) = -2q + 4 - 12q^2 = 0$, $\therefore q_1 = -\frac{2}{3}$ (round) $q_2 = \frac{1}{2}$

$\therefore$ In interval (0,1), maximum values for q is $f\left(\frac{1}{2}\right) = \frac{9}{4}$. If and only if $q = \frac{1}{2}$,

$\therefore \frac{9}{4} \geq \frac{1}{1+4q} \times \frac{1}{1-q^2}$, $\therefore (1 + 4q) \times \left(1 - q^2\right) \leq \frac{9}{4}$, $\therefore \sum\limits_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} \leq \frac{9}{4}$.

When $q = \frac{1}{2}$ $\sum\limits_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} = \frac{9}{4}$. At this time $x_n = \frac{1}{2^{n-1}}$.

Chen Jianan, Lee Xueliang, Guo Meiqun, Feng Xiaoling, kong Huimin and Zheng Nanmin

Fourth solution. By the Cauchy-Schwarz-inequality

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} \cdot \sum_{n=1}^{\infty} x_n \left(x_n + 4x_{n+1}\right) =$$

$$= \sum_{n=1}^{\infty} x_n^2 \frac{x_n}{x_n + 4x_{n+1}} \cdot \sum_{n=1}^{\infty} x_n^2 \frac{x_n + 4x_{n+1}}{x_n} \geq \left(\sum_{n=1}^{\infty} x_n^2\right)^2$$

Again, by the Cauchy-Schwarz-inequality

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} \geq \frac{\left(\sum\limits_{n=1}^{\infty} x_n^2\right)^2}{\sum\limits_{n=1}^{\infty} x_n \left(x_n + 4x_{n+1}\right)} = \frac{\left(\sum\limits_{n=1}^{\infty} x_n^2\right)^2}{\sum\limits_{n=1}^{\infty} x_n^2 + 4 \sum\limits_{n=1}^{\infty} x_n x_{n+1}} \geq$$

$$\geq \frac{\left(\sum\limits_{n=1}^{\infty} x_n^2\right)^2}{\sum\limits_{n=1}^{\infty} x_n^2 + 4\sqrt{\sum\limits_{n=1}^{\infty} x_n^2 \cdot \sum\limits_{n=1}^{\infty} x_{n+1}^2}} = \frac{s^2}{s + 4\sqrt{s\,(s-1)}}$$

where $s = \sum\limits_{n=1}^{\infty} x_n^2$.

The inequality $\frac{s^2}{s+4\sqrt{s(s-1)}} \geq \frac{4}{9}$ is equivalent to each of

$$\left(s^2 - \frac{4}{9}s\right)^2 \geq \frac{256}{81} s\,(s-1) \ \text{ and } \ \frac{s\,(3s-4)^2\,(16+9s)}{81} \geq 0,$$

which obviously holds true. So $\sum\limits_{n=1}^{\infty} \frac{x_n^3}{x_n+4x_{n+1}} \geq \frac{4}{9}$.

Equality holds true only if $\frac{x_n+4x_{n+1}}{x_n}$ is constant and $s = \frac{4}{3}$, which means that $\frac{x_{n+1}}{x_n} = c$ for some constant c implying $x_n = c^{n-1}$ and $s = \sum\limits_{n=1}^{\infty} c^{n-2} = \frac{1}{1-c^2} = \frac{4}{3}$. So $c = \frac{1}{2}$ and $x_n = \frac{1}{2^{n-1}}$, $n = 1, 2, \ldots$

Indeed,

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} = \sum_{n=0}^{\infty} \frac{\frac{1}{8^n}}{\frac{1}{2^n} + \frac{2}{2^n}} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{4}{9}.$$

Albert Stadler

W7. Solution by the proposer. Let $F := [ABC]$. Since $[A_1CB_1I] = [A_1AB] - [IAB_1]$ and $\frac{CA_1}{BC} = \frac{b}{b+c}, \frac{AB_1}{AC} = \frac{c}{a+c}, \frac{IB_1}{BB_1} = = \frac{b}{a+b+c}$ then

$[A_1AC] = \frac{bF}{b+c}, [ABB_1] = \frac{cF}{a+c}, [IAB_1] = \frac{b[ABB_1]}{a+b+c} = \frac{bcF}{(a+b+c)(a+c)},$

$$[A_1CB_1I] = \frac{bF}{b+c} - \frac{bcF}{(a+b+c)(a+c)} = \frac{bF\left((a+c)^2 + ab + bc - bc - c^2\right)}{(a+b+c)(a+c)(b+c)} =$$

$$= \frac{Fab(2c+a+b)}{(a+b+c)(a+c)(b+c)} = \frac{F\left((a+b)^2 - c^2\right)(2c+a+b)}{2(a+b+c)(a+c)(b+c)} =$$

$$= \frac{F(a+b-c)(2c+a+b)}{2(a+c)(b+c)}.$$

Thus

$$\frac{[A_1CB_1I]}{F} = \frac{(a+b-c)(2c+a+b)}{2(a+c)(b+c)}.$$

Now we will find $\max \frac{(a+b-c)(2c+a+b)}{2(a+c)(b+c)}$.

Due to homogeneity of $\frac{(a+b-c)(2c+a+b)}{(a+c)(b+c)}$ we can assume that $c = 1$.

Since $a^2 + b^2 = 1$ then, denoting $t := a+b$ obtain that $t \le \sqrt{2}$ $(\iff a+b \le \sqrt{2(a^2+b^2)})$,

$$(a+1)(b+1) = 1 + t + \frac{t^2-1}{2} = \frac{(t+1)^2}{2}, \frac{(a+b-c)(2c+a+b)}{2(a+c)(b+c)} =$$

$$= \frac{(t-1)(2+t)}{2(a+1)(b+1)} = \frac{(t-1)(2+t)}{(t+1)^2}$$

and, therefore,

$$\max\left(\frac{(a+b-c)(2c+a+b)}{2(a+c)(b+c)}\right) = \max_{0<t\le\sqrt{2}}\left(\frac{(t-1)(t+2)}{(t+1)^2}\right).$$

Since $t+1 \le \sqrt{2}+1 \iff \frac{1}{t+1} \ge \sqrt{2}-1$ then

$$\frac{(t-1)(t+2)}{(t+1)^2} = \frac{t^2+t-2}{(t+1)^2} = 1 - \frac{1}{t+1} -$$

$$-\frac{2}{(t+1)^2} \le 1 - \left(\sqrt{2}-1\right) - 2\left(\sqrt{2}-1\right)^2 = 2 - \sqrt{2} - 6 + 4\sqrt{2} = 3\sqrt{2} - 4$$

and equaliy occurs iff $t = \sqrt{2}$. Thus, max $\dfrac{[A_1CB_1I]}{F} = 3\sqrt{2} - 4$ and can be attained only iff $a = b = \dfrac{\sqrt{2}c}{2}$ because

$$\begin{cases} a^2 + b^2 = c^2 \\ a + b = \sqrt{2}c \end{cases} \iff \begin{cases} a^2 + b^2 = c^2 \\ a = b \end{cases}.$$

Second solution. $2\alpha + 2\beta = \frac{\pi}{2}$, $\alpha + \beta = \frac{\pi}{4}$, $a^2 + b^2 = c^2$
In the $\triangle ABI$, set $BI = x$, $AI = y$, set $\measuredangle BAI = \measuredangle\alpha$, $\measuredangle ABI = \measuredangle\beta$,

$$\frac{x}{\sin\alpha} = \frac{y}{\sin\beta} = \frac{c}{\sin(\pi - \alpha - \beta)} = \frac{c}{\sin(\alpha + \beta)}$$

$\therefore x = \sqrt{2}\sin\alpha \times c$, $y = \sqrt{2}\sin\beta\times$
$S_{\triangle ABI} = \frac{1}{2} \times 2\sin\alpha\sin\beta c^2 \times \sin(\pi - \alpha - \beta) - \frac{2}{\sqrt{2}}\sin\alpha\sin\beta c^2\times$
Similarly avaible $S_{\triangle ABA_1} = \frac{1}{2}c^2 \times \frac{\sin\alpha\sin 2\beta}{\sin(\alpha+2\beta)}$, $S_{\triangle ABB_1} = \frac{1}{2}c^2 \times \frac{\sin 2\alpha\sin\beta}{\sin(2\alpha+\beta)}$
Let S be A_1CB_1I except the quadrilateral area,

$$\therefore S = \frac{1}{2}c^2 \times \left[\frac{2\sin\alpha\sin\beta}{\sin(2\alpha + \beta)} + \frac{\sin\alpha\sin 2\beta}{\sin(\alpha + 2\beta)} - \sqrt{2}\sin\alpha\sin\beta\right],$$

$S_{\triangle ABC} = \frac{1}{2}c^2 \times \sin 2\alpha\sin 2\beta$.
Now to make the minimum $\frac{S}{S_{\triangle ABC}}$, so the proportion of quadrilateral is maximized

$$\frac{S}{S_{\triangle ABC}} = \frac{\sin\beta}{\sin(2\alpha + \beta)\sin 2\beta} + \frac{\sin\alpha}{\sin(\alpha + 2\beta)\sin 2\alpha} - \frac{\sqrt{2}\sin\alpha\sin\beta}{\sin 2\alpha\sin 2\beta} =$$

$$= \frac{1}{2\sin(2\alpha + \beta)\cos\beta} + \frac{1}{2\sin(\alpha + 2\beta)\cos\alpha} - \frac{\sqrt{2}}{4\cos\alpha\cos\beta} =$$

$$= \frac{1}{2\cos\alpha\cos\beta} + \frac{1}{2\cos\beta\cos\alpha} - \frac{\frac{\sqrt{2}}{2}}{2\cos\alpha\cos\beta} = \frac{2 - \frac{\sqrt{2}}{2}}{2\cos\alpha\cos\beta}$$

Now to make the $f(\alpha) = 2\cos\alpha\cos\beta$ max, $\alpha + \beta = \frac{\pi}{4}$,

$$\therefore f(\alpha) = 2\cos\alpha\cos\left(\frac{\pi}{4} - \alpha\right) = 2\cos\left(\frac{\sqrt{2}}{2}\cos\alpha + \frac{\sqrt{2}}{2}\sin\alpha\right) =$$

$$= \sqrt{2}\cos^2\alpha + \sqrt{2}\sin\alpha\cos\alpha = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(2\cos^2\alpha - 1\right) + \frac{\sqrt{2}}{2}\sin 2\alpha =$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cos 2\alpha + \frac{\sqrt{2}}{2}\sin 2\alpha = \frac{\sqrt{2}}{2} + \sin\left(2\alpha + \frac{\pi}{4}\right)$$

$\max f(\alpha) = \frac{\sqrt{2}}{2} + 1$, at this time $\alpha = \frac{\pi}{8}$ min $\frac{S}{S_{\triangle ABC}} = \frac{2 - \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2} + 1} = 5 - 3\sqrt{2}$.

So $\frac{S_{[A_1CB_1I]}}{S_{\triangle ABC}} = 1 - \frac{S}{S_{\triangle ABC}} = 1 - \left(5 - 3\sqrt{2}\right) = 3\sqrt{2} - 4.$

Chen Jianan, Lee Xueliang, Guo Meiqun, Feng Xiaoling, kong Huimin and Zheng Nanmin

Third solution. We can assume that $AC = 1$. We put $\alpha = \angle BAC = (=$ angle at $A)$. Then $area\,(\triangle ABC) = \frac{1}{2}\tan(\alpha)$, $area\,(\triangle AA_1C) = \frac{1}{2}\tan\left|\frac{\alpha}{2}\right|$, $area\,(\triangle BCB_1) = \frac{1}{2}\tan^2(\alpha)\tan\left|\frac{\frac{\pi}{2}-\alpha}{2}\right|$,

$$area\,(\triangle ABI) = \frac{1}{2}AI\cdot BI\sin\left|\pi - \frac{\alpha}{2} - \frac{\frac{\pi}{2}-\alpha}{2}\right| = \frac{1}{2}AI\cdot BI\frac{\sqrt{2}}{2} =$$

$$= \frac{1}{2}\frac{AB\sin\left|\frac{\frac{\pi}{2}-\alpha}{2}\right|}{\sin\left|\frac{3\pi}{4}\right|}\cdot\frac{AB\sin\left|\frac{\alpha}{2}\right|}{\sin\left|\frac{3\pi}{4}\right|}\sin\left|\frac{3\pi}{4}\right| =$$

$$= \frac{\sqrt{2}}{2}AB^2\sin\left|\frac{\frac{\pi}{2}-\alpha}{2}\right|\sin\left|\frac{\alpha}{2}\right| = \frac{\sqrt{2}}{2}\frac{\sin\left|\frac{\frac{\pi}{2}-\alpha}{2}\right|\sin\left|\frac{\alpha}{2}\right|}{\cos^2\alpha} = \frac{\left|\cos\left|\frac{\alpha}{2}\right| - \sin\left|\frac{\alpha}{2}\right|\right|\sin\left|\frac{\alpha}{2}\right|}{2\cos^2(\alpha)},$$

$area\,(\diamond A_1CB_1I) = area\,(\triangle AA_1C) + area\,(\triangle BCB_1) + area\,(\triangle ABI) - area\,(\triangle ABC)$. Then

$$\frac{area\,(\diamond A_1CB_1I)}{area\,(\triangle ABC)} =$$

$$= \frac{\frac{1}{2}\tan\left|\frac{\alpha}{2}\right| + \frac{1}{2}\tan^2(\alpha)\tan\left|\frac{\frac{\pi}{2}-\alpha}{2}\right| + \frac{\left|\cos\left|\frac{\alpha}{2}\right|-\sin\left|\frac{\alpha}{2}\right|\right|\sin\left|\frac{\alpha}{2}\right|}{2\cos^2(\alpha)} - \frac{1}{2}\tan(\alpha)}{\frac{1}{2}\tan(\alpha)} =$$

$$= \frac{\tan\left|\frac{\alpha}{2}\right|}{\tan(\alpha)} + \tan(\alpha)\frac{1-\tan\left|\frac{\alpha}{2}\right|}{1+\tan\left|\frac{\alpha}{2}\right|} + \frac{\left|\cos\left|\frac{\alpha}{2}\right| - \sin\left|\frac{\alpha}{2}\right|\right|\sin\left|\frac{\alpha}{2}\right|}{\sin(\alpha)\cos(\alpha)} - 1 =$$

$$= \frac{1-\tan^2\left|\frac{\alpha}{2}\right|}{2} + \frac{2\tan\left|\frac{\alpha}{2}\right|}{2\cos(\alpha)} - 1 = \frac{1-\tan^2\left|\frac{\alpha}{2}\right|}{2} + \frac{2\tan\left|\frac{\alpha}{2}\right|}{\left|1+\tan\left|\frac{\alpha}{2}\right|\right|^2} + \frac{1-\tan\left|\frac{\alpha}{2}\right|}{1-\tan^2\left|\frac{\alpha}{2}\right|} - 1 =$$

$$\overset{u=\tan\left|\frac{\alpha}{2}\right|}{=} \frac{1-u^2}{2} + \frac{2u}{(1+u)^2} + \frac{1+u^2}{2(1+u)} - 1 = f(u)$$

We have

$$\frac{d}{du}\left|\frac{1-u^2}{2} + \frac{2u}{(1+u)^2} + \frac{1+u^2}{2(1+u)} - 1\right| = \frac{\left(1-2u-u^2\right)\left(3+u+2u^2\right)}{2(1+u)^3}$$

So $f(u)$ gets minimal at the positive zero of $1 - 2u - u^2 = 0$ or at $u = \sqrt{2} - 1$. If $\tan\left(\frac{\alpha}{2}\right) = \sqrt{2} - 1$ then $\tan(\alpha) = 1$.

So $\alpha = \frac{\pi}{4}$ and thus the right triangle with the greatest ratio is the isosceles right triangle.

Albert Stadler

Fourth solution. We use the usual notations. If r is the inradius, then $r = \frac{a+b-c}{2}$. By bisector theorem we obtain that $A_1C = \frac{ab}{b+c}$ and $B_1C = \frac{ab}{a+c}$.
Yields that

$$2\left[A_1CB_1I\right] = 2\left[A_1CI\right] + 2\left[CB_1I\right] = r\left(\frac{ab}{b+c} + \frac{ab}{a+c}\right)$$

so

$$\frac{\left[A_1CB_1I\right]}{[ABC]} = \frac{a+b-c}{2} \cdot \frac{a+b+2c}{(a+c)(b+c)}$$

Because we suspect that the maximum value of this ratio is reached within an isosceles right triangle, we demonstrate that

$$\frac{(a+b-c)(a+b+2c)}{(a+c)} \leq \frac{2\sqrt{2}}{3+2\sqrt{2}}$$

which yields that the greatest value is $\frac{\sqrt{2}}{3+2\sqrt{2}}$.
We have:

$$\frac{(a+b-c)(a+b+2c)}{(a+c)(b+c)} \leq \frac{2\sqrt{2}}{3+2\sqrt{2}} \Leftrightarrow$$

$$\Leftrightarrow \left(3+2\sqrt{2}\right)c(a+b) - 2\sqrt{2}c(a+b) \leq 2\sqrt{2}\left(a^2+b^2+ab\right) + \left(3+2\sqrt{2}\right)\left(a-2ab+b^2\right) \Leftrightarrow$$

$$\Leftrightarrow 3c(a+b) \leq 2\sqrt{2}\left(2a^2-ab+2b^2\right) + 3\left(a^2-2ab+b^2\right)$$

and squared we obtain

$$9(a+b)^2\left(a^2+b^2\right) \leq 8\left(2a^2-ab+2b^2\right)^2 + 9\left(a^2-2ab+b^2\right)^2 +$$

$$+12\sqrt{2}\left(2a^2-ab+2b^2\right)\left(a^2-2ab+b^2\right) \Leftrightarrow$$

$$\Leftrightarrow 32a^4 - 86a^3b + 108a^2b^2 - 86ab^3 + 32b^4 + 24\sqrt{2}a^4 - 60\sqrt{2}a^3b + 72\sqrt{2}a^2b^2 -$$

$$-60\sqrt{2}ab^3 + 24\sqrt{2}b^4 \geq 0 \Leftrightarrow (a-b)^2\left(16a^2-11ab+16b^2\right) + 6\sqrt{2}(a-b)^2\left(2a^2-ab+2b^2\right) \geq 0,$$

true, since $16a^2 - 11ab + 16b^2 0$, $2a^2 - ab + 2b^2 > 0$. We have equality iff $a = b$, i.e the triangle is isosceles right triangle.

Neculai Stanciu

W8. Solution by the proposer. Note that $\Delta\left(x^2, y^2, z^2\right) = (x+y+z)(x+y-z)(x-y+z)(-x+y+z)$ and for positive x, y, z we have equivalency

$$\Delta\left(x^2, y^2, z^2\right) > 0 \iff \begin{cases} x+y>z \\ y+z>x \\ z+x>y \end{cases}.$$

Due symmetry and homogeneity of $\Delta\left(a^n, b^n, c^n\right) > 0$ WLOG we assume that $a \geq b \geq 1$.
Then for any $n \in \mathbb{N}$ we have

$$\begin{cases} \Delta\left(a^{2n}, b^{2n}, c^{2n}\right) > 0 \\ a \geq b \geq c = 1 \end{cases} \iff \begin{cases} b^n + 1 > a^n \\ a \geq b \geq c = 1 \end{cases}.$$

Suppose that $a > b$, then

$$a^n = (b + (a-b))^n > b^n + n(a-b)b^{n-1} > b^n + n(a-b) > b^n + 1$$

for any $n > \dfrac{1}{a-b}$. It is contradict to $b^n + 1 > a^n$ which holds for any $n \in \mathbb{N}$.
Thus $a = b$ and, therefore, triangle should be isosceles with two equal sides, which not less then third one.
Let now $a = b \geq c$ then

$$\Delta\left(a^n, b^n, c^n\right) = 2a^n b^n + 2b^n c^n + 2c^n a^n - a^{2n} - b^{2n} - c^{2n} =$$

$$= 4c^n a^n - c^{2n} \geq 3c^{2n} > 0.$$

Second solution. 1).When $a = b = c$, $V\left(a^n, b^n, c^n\right) = V\left(a^n, a^n, a^n\right) = 2a^n a^n + 2a^n a^n + 2a^n a^n - (a^n)^2 - (a^n)^2 - (a^n)^2 = 6a^{2n} - 3a^{2n} = 3a^{2n} > 0$.

2). When $b = c > a$,
$V\left(a^n, b^n, c^n\right) = V\left(a^n, b^n, b^n\right) = 2a^n b^n + 2b^n b^n + 2d^n b^n - (a^n)^2 - (b^n)^2 - (b^n)^2 = 2b^{2n} + 4a^n b^n - a^{2n} - 2b^{2n} = 4a^n b^n - a^{2n} = a^n\left(4b^n - a^n\right)$, $Qb > a$, $\therefore 4b^n > a^n$, $\therefore a^n\left(4b^n - a^n\right) > 0$, $\therefore \left(a^n, b^n, c^n\right) > 0$.

Chen Jianan, Lee Xueliang, Guo Meiqun, Feng Xiaoling, kong Huimin and Zheng Nanmin

Third solution. By symmetry we can assume that $a \le b \le c$. We distinguish two cases:
a). $b < c$

$$\triangle\left(a^n, b^n, c^n\right) = -\left(a^n - b^n\right)^2 + 2c^n\left(a^n + b^n\right) - c^{2n} \le 2c^n\left(a^n + b^n\right) - c^{2n} =$$

$$= c^{2n}\left(-1 + 2\frac{a^n + b^n}{c^n}\right) < 0$$

if n is sufficiently big, since $\frac{a}{c-1} < 1$, $\frac{b}{c} < 1$.
b). $b = c$

$$\triangle\left(a^n, b^n, b^n\right) = 4a^nb^n - a^{2n} = a^n\left(4b^n - a^n\right) > 0,$$

for all natural numbers n. So $\triangle\left(a^n, b^n, c^n\right) > 0$ for any natural number n if and only if a, b, c form the sides of an isosceles triangle whose third side is smaller than the two legs or a, b, c form the sides of an equilateral triangle.

Albert Stadle

W9. Solution by the proposer. a) Follows immediately from inequality $s^2 \le 4R^2 + 4Rr + 3r^2$. Really,

$$R^2 - 4r^2 \ge \frac{1}{5}\cdot\left(s^2 - 27r^2\right) \iff s^2 \le 5R^2 + 7r^2 \iff$$

$$0 \le \left(4R^2 + 4Rr + 3r^2 - s^2\right) + \left(R - 2r\right)^2.$$

b) Recall ([1]), that a triple (R, r, s) of positive real numbers can determine a triangle, where R, r, and s be a circumradius, inradius and semiperimeter respectively iff $(R, r, s) \in \bar{\Delta} := \{(R, r, s) \mid R \ge 2r \text{ and } L(R, r) \le s^2 \le M(R, r)\}$ where

$$L(R, r) = 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R(R - 2r)}$$

and

$$M(R, r) = 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R(R - 2r)}.$$

Since a triangle is equilateral iff $R = 2r$ then set

$$\Delta := \left\{(R, r, s) \mid (R, r, s) \in \bar{\Delta} \text{ and } R \ne 2r\right\}$$

determine all non-equilateral triangles.
Thus,

$$\max K = \min_{(R,r,s)\in\Delta} \frac{R^2 - 4r^2}{s^2 - 27r^2} = \min_{R>2r}\left(\min_{s} \frac{R^2 - 4r^2}{s^2 - 27r^2}\right) = \min_{R>2r} \frac{R^2 - 4r^2}{M(R, r) - 27r^2} =$$

$$= \min_{R>2r} \frac{R^2-4r^2}{2R^2+10Rr-28r^2+2(R-2r)\sqrt{R(R-2r)}} = \min_{R>2r} \frac{R+2r}{2R-14r+2\sqrt{R(R-2r)}} =$$

$$= \min_{R>2r} \frac{1+\dfrac{2r}{R}}{2-7\cdot\dfrac{2r}{R}+2\sqrt{1-\dfrac{2r}{R}}}.$$

Denoting $t := \sqrt{1-\dfrac{2r}{R}}$ we obtain that $t \in (0,1)$ and, therefore,

$$K_* := \max K = \min_{t\in(0,1)} \frac{2-t^2}{9+2t-7t^2} = \frac{23+\sqrt{17}}{8},$$

because $\dfrac{23+\sqrt{17}}{128}$ is smallest real k for which equation $\dfrac{2-t^2}{9+2t-7t^2} = k$ have solution in $(0,1)$.

Indeed, if equation $\dfrac{2-t^2}{9+2t-7t^2} = k$ have solution then

$$2-t^2 = k\left(9+2t-7t^2\right) \iff (7k-1)\,t^2-2kt-9k+2=0$$

yields

$$k^2+(7k-1)(9k-2) = 64k^2-23k+2 \geq 0 \implies k \geq \frac{23+\sqrt{17}}{128}.$$

Since for $k_* := \dfrac{23+\sqrt{17}}{128}$ equation

$$\frac{2-t^2}{9+2t-7t^2} = k_*$$

have only solution

$$t_* = \frac{k_*}{7k_*-1} = \frac{23+\sqrt{17}}{33+7\sqrt{17}} = \frac{5-\sqrt{17}}{2} \in (0,1)$$

then

$$K_* = \min_{t\in(0,1)} \frac{2-t^2}{9+2t-7t^2} = \frac{2-t_*^2}{9+2t_*-7t_*^2} = \frac{23+\sqrt{17}}{128}.$$

So, for any triangle holds inequality

$$R^2-4r^2 \geq \frac{23+\sqrt{17}}{128}\left(s^2-27r^2\right)$$

and not exist constant $K > \dfrac{23+\sqrt{17}}{128}$ which provide inequality in (b).

c) $\lim\limits_{R\to 2r}\dfrac{R^2-4r^2}{s^2-27r^2}=\dfrac{2}{9}$ because

$$\frac{R^2-4r^2}{2R^2+10Rr-28r^2+2\left(R-2r\right)\sqrt{R\left(R-2r\right)}}\leq$$

$$\leq\frac{R^2-4r^2}{s^2-27r^2}\leq\frac{R^2-4r^2}{2R^2+10Rr-28r^2-2\left(R-2r\right)\sqrt{R\left(R-2r\right)}}\iff$$

$$\frac{R+2r}{2R+14r+2\sqrt{R\left(R-2r\right)}}\leq\frac{R^2-4r^2}{s^2-27r^2}\leq\frac{R+2r}{2R+14r-2\sqrt{R\left(R-2r\right)}}\text{ and}$$

$$\lim_{R\to 2r}\frac{R+2r}{2R+14r+2\sqrt{R\left(R-2r\right)}}=\lim_{R\to 2r}\frac{R+2r}{2R+14r-2\sqrt{R\left(R-2r\right)}}=\frac{2}{9}.$$

[1] D.S.Mitrinovic,J.E.Pecaric,V.Volnec.Recent Advances In Geometric Inequalities.

Second solution. Let a,b and c be the sides of the triangle, and let $\triangle$ be its area. Then

$$s=\frac{a+b+c}{2},\triangle=\sqrt{s\left(s-a\right)\left(s-b\right)s-c},R=\frac{abc}{4\triangle},r=\frac{\triangle}{s}$$

Let $x=b+c-a,y=c+a-b,z=a+b-c$. Then $x\geq 0,y\geq 0,z\geq 0$, $x+y=2c$, $y+z=2a$, $z+x=2b$, $x+y+z=a+b+c$. We express s,R,r in terms of $x,y,z:$ $s=\frac{x+y+z}{2}$, $\triangle=\frac{1}{4}\sqrt{xyz\left(x+y+z\right)}$, $R=\frac{abc}{4\triangle}=\frac{(x+y)(y+z)(z+x)}{8\sqrt{xyz(x+y+z)}}$, $r=\frac{\triangle}{s}=\frac{1}{2}\sqrt{\frac{xyz}{x+y+z}}$

Then

$$f\left(x,y,z\right):=\frac{R^2-4r^2}{s^2-27r^2}=\frac{\frac{(x+y)^2(y+z)^2(z+x)^2}{64xyz(x+y+z)}-\frac{xyz}{x+y+z}}{\left(\frac{x+y+z}{2}\right)^2-\frac{27xyz}{4(x+y+z)}}=$$

$$=\frac{\left(x+y\right)^2\left(y+z\right)^2\left(z+x\right)^2-64\left(xyz\right)^2}{17xyz\left(\left(x+y+z\right)^3-27xyz\right)}=$$

$$=\frac{\left(\left(x+y\right)\left(y+z\right)\left(z+x\right)-8xyz\right)\left(\left(x+y\right)\left(y+z\right)\left(z+x\right)+8xyz\right)}{16xyz\left(\left(x+y+z\right)^3-27xyz\right)}$$

$f\left(x,y,z\right)$ is a symmetric function in x,y,z. We can assume without loss of generality that $x\leq y\leq z$. We claim that

$$f\left(x,y,z\right)\geq f\left(\frac{x+y}{2},\frac{x+y}{2},z\right)\text{ if }z\geq\frac{x+y}{2}$$

Indeed

$$f\left(\frac{x+y}{2},\frac{x+y}{2},z\right)=\frac{\left(x+y\right)^2\left(\frac{x+y}{2}+z\right)^4-4\left(x+y\right)^4z^2}{4\left(x+y\right)^2z\left(\left(x+y+z\right)^3-\frac{27}{4}\left(x+y\right)^2z\right)}=$$

$$= \frac{\left(\frac{x+y}{2}+z\right)^4 - 4(x+y)^2 z^2}{z\left(4(x+y+z)^3 - 27(x+y)^3 z\right)} =$$

$$= \frac{\left(\left(\frac{x+y}{2}+z\right)^2 - 2(x+y)z\right)\left(\left(\frac{x+y}{2}+z\right)^2 + 2(x+y)z\right)}{z(x+y-2z)^2(4x+4y+z)} =$$

$$= \frac{(x+y)^2 + 12(x+y)z + 4z^2}{16z(4(x+y)+z)}$$

$$f(x,y,z) - f\left(\frac{x+y}{2}, \frac{x+y}{2}, z\right) = \frac{(x+y)^2(y+z)^2(z+x)^2 - 64(xyz)^2}{16xyz\left((x+y+z)^2 - 27xyz\right)} -$$

$$-\frac{(x+y)^2 + 12(x+y)z + 4z^2}{16z(4(x+y+z))} =$$

$$= \frac{(x-y)^2(x+y+z)\left(z^4 + 5(x+y)z^3 + (4x^2 - 3xy + 4y^2)z^2 - 6xy(x+y)z - xy(x+y)^2\right)}{16xyz\left((x+y+z)^3 - 27xyz\right)(4x+4y+z)}$$

as is easily verified. By the AM-GM inequality $(x+y+z)^3 - 27xyz \geq 0$. It is sufficient to prove that

$$g(x,y,z) := z^4 + 5(x+y)z^3 + (4x^2 - 3xy + 4y^2)z^2 - 6xy(x+y)z - xy(x+y)^2 \geq 0 \tag{2}$$

for $z \geq \frac{x+y}{2}$.

We note that

$$g\left(x, y, \frac{x+y}{2}\right) = \frac{27}{16}(x^2 - y^2)^2 \geq 0$$

Furthermore

$$\frac{\theta}{\theta z} g(x,y,z) = 4z^3 + 15(x+y)z^2 + 2(4x^2 - 3xy + 4y^2)z - 6xy(x+y) \geq$$

$$\geq 4\left(\frac{x+y}{2}\right)^3 + 15(x+y)\left(\frac{x+y}{2}\right)^2 + 2(4x^2 - 3xy + 4y^2)\left(\frac{x+y}{2}\right) - 6xy(x+y) =$$

$$= \frac{1}{4}(x+y)(33x^2 - 2xy + 33y^2) \geq 0 \text{ for } z \geq \frac{x+y}{2}.$$

So

$$g(x,y,z) = g\left(x,y,\frac{x+y}{2}\right) + \int_{\frac{x+y}{2}}^{z} \frac{\theta}{\theta t} g(x,y,t)\,dt \geq 0$$

and (2) and thus (1) follows.
We conclude that

$$\min_{x,y,z>0} f(x,y,z) = \min_{t,z>0} f(t,t,z) = \min_{x,z>0} \frac{(2t^2) + 12(2t)z + 4z^2}{16z(4(2t)+z)} =$$

$$= \min_{t,z>0} \frac{t^2 + 6tz + z^2}{4z(8t+z)} = \min_{z>0} \frac{1+6z+z^2}{4z(8+z)}$$

We have

$$\frac{d}{dz} \cdot \frac{1+6z+z^2}{4z(8+z)} = \frac{z^2 - z - 4}{2z^2(8+z)^2}$$

So the minimum is assumed for $z = \frac{1+\sqrt{17}}{2}$ so that

$$\min_{x,y,z>0} f(x,y,z) = \min_{z>0} \frac{1+6z+z^2}{4z(8+z)} = \lim_{z\to\frac{1+\sqrt{17}}{2}} \frac{1+6z+z^2}{4z(8+z)} = \frac{23+\sqrt{17}}{128} \approx 0,211899 > 0.2$$

So the maximum value for the constant K equals $\frac{23+\sqrt{17}}{128}$. This proves a) and b). We have, by the AM-GM inequality,

$$R = \frac{(x+y)(y+z)(z+x)}{8\sqrt{xyz(x+y+z)}} \geq \frac{2\sqrt{xy}2\sqrt{yz}2\sqrt{zx}}{8\sqrt{xyz(x+y+z)}} = \frac{\sqrt{xyz}}{\sqrt{(x+y+z)}} = 2r$$

with equality if and only if $x = y = z$. So, by (1),

$$\lim_{R\to 2r} \frac{R^2 - 4r^2}{s^2 - 27r^2} = \lim_{z\to t} (t,t,z) = \lim_{z\to t} \frac{t^2+6tz+z^2}{4z(8t+z)} = \frac{2}{9} = 0,\overline{2}$$

Albert Stadler

Third solution. a). Letting a, b, c be the sides of the triangle, we know that a). $r = \sqrt{(s-a)(s-b)(s-c)/s}$, $R = 2(abc)/(r(a+b+c))$. Moreover we define the classical change $x = b + c - a$ and cyclic. By observing that
$(x+y)(y+z)(z+x) = (x+y+z)(xy+yz+zx) - xyz$, the inequality becomes

$$\frac{((x+y+z)(xy+yz+zx) - xyz)^2}{64(x+y+z)xyz} - \frac{xyz}{x+y+z} \geq \frac{1}{5}\left(\frac{(x+y+z)^2}{4} - \frac{27xyz}{4(x+y+z)}\right)$$

Now we define the new variables

$$x+y+z = 3u, \quad xy+yz+zx = 3v^2, \quad xyz = w^3$$

Trivial AGM yields $w \le v \le u$. The inequality becomes

$$\frac{3}{64}\frac{13w^6 + w^3(-10uv^2 - 48u^3) + 45u^2v^4)}{uw^3} \ge 0$$

or

$$P(w^3) = 13w^6 + w^3(-10uv^2 - 48u^3) + 45u^2v^4 \ge 0$$

The minimum od the parabola $P(w^3)$ occurs at $w_0^3 = \frac{24}{13}u^3 + \frac{5}{13}uv^2$ but this is forbidden because $w^3 \le v^3 \le u^3$. This means that the range of the values of w^3 is contained in $[0, w_0^3)$ and in this interval the parabola decreases. It follows that $P(w^3) \ge 0$ if and only if it holds at the extreme value of w^3 and the standard theory states that this happens when $x = y$ or cyclic. If $x = y$ the inequality becomes

$$\frac{1}{16}\frac{(5y^2 - 2yz + z^2)(y - z)^2}{(2y + z)z} \ge 0$$

which clearly holds.

b). It is equivalent to find the smallest number Q such that

$$Q\frac{((x+y+z)(xy+yz+zx) - xyz)^2}{64(x+y+z)xyz} - \frac{xyz}{x+y+z} \ge \frac{1}{5}\left(\frac{(x+y+z)^2}{4} - \frac{27xyz}{4(x+y+z)}\right)$$

which is equivalent via the change $x = b + c - a$ and cyclic to

$$\frac{1}{64}\frac{9Qu^2v^4 - w^3(48u^3 + 2Quv^2) + w^6(48 - 7Q)}{uw^3} \ge 0$$

which is equivalent to to $P(v^2) = 9Qu^2v^4 - 2Quv^2w^3 - w^3 48u^3 + w^6(48 - 7Q) \ge 0$. The minimum of the parabola $P(v^2)$ occurs at

$$v^2 = \frac{w^3}{9u} \le \frac{v^3}{9v} = \frac{v^2}{9}$$

This means that the range of the parabola is contained in $(0, w^3/(9u))$ and in this range the parabola decreases. Therefore the inequality holds if and only if it holds when v^2 assumes its maximum value. This in turn holds when at least two of the variables x, y, z are equal so we suppose $z = y$ and get

$$\frac{1}{16}(y - z)^2\frac{Qy^2 + 6yQz - 32yz + Qz^2 - 4z^2}{(2y + z)z} \ge 0$$

If $Q \ge 16/3$ every addend of the numerator is nonnegative. If $Q < 16/3$ we need

$$2\sqrt{Q(Q-4)} \ge 6Q - 32 \iff \frac{23 - \sqrt{17}}{4} \le Q \le \frac{23 + \sqrt{17}}{4}$$

and since $16/3 < (23+\sqrt{17})/4$ it follows that the searched number Q is $(23-\sqrt{17})/4$. The greatest K is then $(23+\sqrt{17})/128$

c). The limit is $2/9$. Indeed we have $R = 2r$ if and only if the triangle is equilateral so we let $(x,y,z) \to (p,p,p)$ in

$$F(x,y,z) \doteq \frac{\frac{((x+y+z)(xy+yz+zx)-xyz)^2}{64(x+y+z)xyz} - \frac{xyz}{x+y+z}}{\left(\frac{(x+y+z)^2}{4} - \frac{27xyz}{4(x+y+z)}\right)} - \frac{2}{9} \doteq \frac{F_1(x,y,z)}{F_2(x,y,z)}$$

and make the limit

$$\lim_{(\alpha,\beta,\gamma)\to(0,0,0)} F(p+\alpha, p+\beta, p+\gamma)$$

$$\begin{aligned} &F_1(p+\alpha,p+\beta,p+\gamma) = \\ &= -96p^4(\alpha^3+\beta^3+\gamma^3) - 576p^4\gamma\beta\alpha + 144p^4\sum_{\text{sym}}\alpha^2\beta + O((\alpha^2+\beta^2+\gamma^2)^2) \\ &F_2(p+\alpha,p+\beta,p+\gamma) = \\ &= 3888p^5\sum_{\text{cyc}}(\alpha^2-\alpha\beta) + 5616p^4\sum_{\text{cyc}}\alpha^3 + 1296p^4\sum_{\text{sym}}\alpha^2\beta - 24624p^4\alpha\beta\gamma + \\ &+O((\alpha^2+\beta^2+\gamma^2)^2) \end{aligned}$$

The linear change of coordinates

$$r = \frac{\alpha}{\sqrt{3}} + \frac{\beta}{\sqrt{2}} + \frac{\gamma}{\sqrt{6}}, \quad s = \frac{\alpha}{\sqrt{3}} - \frac{\beta}{\sqrt{2}} + \frac{\gamma}{\sqrt{6}}, \quad t = \frac{\alpha}{\sqrt{3}} - \frac{2}{\sqrt{6}}\gamma,$$

sets F_1 and F_2 as functions of (r,s,t) as (it diagonalizes the leading quadratic term of F_2)

$$F_1 \to 72p^4\sqrt{6}t^3 - 216p^4s^2t\sqrt{6} + O((r^2+s^2+t^2)^2)$$

$$F_2 \to 5832p^5(s^2+t^2) + 9720\sqrt{3}p^4r(s^2+t^2) - 1944p^4s^2\sqrt{6}t + 648p^4t^3\sqrt{6} + O((r^2+s^2+t^2)^2)$$

We observe that $|F_1| \le Ct(s^2+t^2)$ while $F_2 = 5832p^5(s^2+t^2) + o(r^2+s^2+t^2)$ and then the limit equals zero proving the result.

Paolo Perfetti

W10. Solution by the proposer. Evidently

$$a^2+b^2+c^2+2abc = 1 \iff \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} + 2 = \frac{1}{abc}$$

If we call $\frac{a}{bc} = x$, $\frac{b}{ca} = y$, $\frac{c}{ab} = z$, we can read the constraint as

$$x+y+z+2 = xyz$$