W59-Inequality in Inner Product Space.

Problem with a solution proposed by Arkady Alt, San Jose, California, USA.

Let $\mathbf{E}$ be a Inner Product Space with dot product $- \cdot -$ and $\mathbf{F}$ be proper nonzero subspace. Let $P : \mathbf{E} \to \mathbf{E}$ be orthogonal projection $\mathbf{E}$ on $\mathbf{F}$.

a) Prove that for any $\mathbf{x}, \mathbf{y} \in \mathbf{E}$, holds inequality

$$|\mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot P(\mathbf{y}) - \mathbf{y} \cdot P(\mathbf{x})| \leq \|\mathbf{x}\| \|\mathbf{y}\|;$$

b) Determine all cases when equality occurs.

Solution.

a)

First recall some properties of $\mathbf{P}$.

By condition $\mathbf{x} - P(\mathbf{x}) \perp \mathbf{F}$ for any $\mathbf{x} \in \mathbf{E}$, there is $(\mathbf{x} - P(\mathbf{x})) \cdot \mathbf{y} = 0$ for any $\mathbf{y} \in \mathbf{F}$.

Then:

1. For any $\mathbf{x} \in \mathbf{F}$ we have $P(\mathbf{x}) = \mathbf{x}$ because $P(\mathbf{x}) \in \mathbf{F}$, $\mathbf{x} - P(\mathbf{x}) \in \mathbf{F} \Rightarrow$ $(\mathbf{x} - P(\mathbf{x})) \cdot (\mathbf{x} - P(\mathbf{x})) = 0 \Leftrightarrow P(\mathbf{x}) = \mathbf{x}$.

2. $\mathbf{x} \cdot P(\mathbf{y}) = \mathbf{y} \cdot P(\mathbf{x})$, for any $\mathbf{x}, \mathbf{y} \in \mathbf{E}$.

(Indeed, $(\mathbf{x} - P(\mathbf{x})) \cdot (\mathbf{y} - P(\mathbf{y})) = (\mathbf{x} - P(\mathbf{x})) \cdot \mathbf{y} - (\mathbf{x} - P(\mathbf{x})) \cdot P(\mathbf{y}) =$ $\mathbf{x} \cdot \mathbf{y} - P(\mathbf{x}) \cdot \mathbf{y}$ and $(\mathbf{x} - P(\mathbf{x})) \cdot (\mathbf{y} - P(\mathbf{y})) = \mathbf{x} \cdot (\mathbf{y} - P(\mathbf{y})) - P(\mathbf{x}) \cdot (\mathbf{y} - P(\mathbf{y})) =$ $\mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot P(\mathbf{y}) \Rightarrow \mathbf{x} \cdot \mathbf{y} - P(\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot P(\mathbf{y}) \Leftrightarrow \mathbf{x} \cdot P(\mathbf{y}) = \mathbf{y} \cdot P(\mathbf{x})$.

3. $\|\mathbf{x}\|^2 = \|P(\mathbf{x})\|^2 + \|\mathbf{x} - P(\mathbf{x})\|^2$, for any $\mathbf{x} \in \mathbf{E}$.

($\|P(\mathbf{x})\|^2 + \|\mathbf{x} - P(\mathbf{x})\|^2 = \|P(\mathbf{x})\|^2 + (\mathbf{x} - P(\mathbf{x})) \cdot (\mathbf{x} - P(\mathbf{x})) = \|\mathbf{x}\|^2 - 2\mathbf{x} \cdot P(\mathbf{x}) +$ $2P(\mathbf{x}) \cdot P(\mathbf{x}) = \|\mathbf{x}\|^2 - 2(\mathbf{x} - P(\mathbf{x})) \cdot P(\mathbf{x}) = \|\mathbf{x}\|^2$.)

4. For any $\mathbf{x} \in \mathbf{E}$ holds $\|\mathbf{x} - 2P(\mathbf{x})\| = \|\mathbf{x}\|$.

Indeed $\|\mathbf{x} - 2P(\mathbf{x})\|^2 = (\mathbf{x} - 2P(\mathbf{x})) \cdot (\mathbf{x} - 2P(\mathbf{x})) = \|\mathbf{x} - P(\mathbf{x})\|^2 - 2P(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{P}(\mathbf{x})) +$ $\|P(\mathbf{x})\|^2 = \|P(\mathbf{x})\|^2 + \|\mathbf{x} - P(\mathbf{x})\|^2 = \|\mathbf{x}\|^2$.

Using property **4**. and Cauchy inequality we get:

$\|\mathbf{x}\| \|\mathbf{y}\| = \|\mathbf{x}\| \cdot \|\mathbf{y} - 2P(\mathbf{y})\| \geq \mathbf{x} \cdot (\mathbf{y} - 2P(\mathbf{y}))$

and

$\|\mathbf{x}\| \|\mathbf{y}\| = \|\mathbf{x} - 2P(\mathbf{x})\| \cdot \|\mathbf{y}\| \geq (\mathbf{x} - 2P(\mathbf{x})) \cdot \mathbf{y}$.

Adding this inequalities we obtain

$2\|\mathbf{x}\| \|\mathbf{y}\| \geq \mathbf{x} \cdot (\mathbf{y} - 2P(\mathbf{y})) + (\mathbf{x} - 2P(\mathbf{x})) \cdot \mathbf{y} \Leftrightarrow$

(**1**) $\quad \|\mathbf{x}\| \|\mathbf{y}\| \geq \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot P(\mathbf{y}) - \mathbf{y} \cdot P(\mathbf{x})$.

By substitution $\mathbf{y} := -\mathbf{y}$ from inequality (**1**) follows inequality

$$\|\mathbf{x}\| \|\mathbf{y}\| \geq -\mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot P(\mathbf{y}) + \mathbf{y} \cdot P(\mathbf{x}) \Leftrightarrow$$

(**2**) $\quad \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot P(\mathbf{y}) - \mathbf{y} \cdot P(\mathbf{x}) \geq -\|\mathbf{x}\| \|\mathbf{y}\|$.

Thus, for any $\mathbf{x}, \mathbf{y} \in \mathbf{E}$ holds inequality

$-\|\mathbf{x}\| \|\mathbf{y}\| \leq \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot P(\mathbf{y}) - \mathbf{y} \cdot P(\mathbf{x}) \leq \|\mathbf{x}\| \|\mathbf{y}\| \Leftrightarrow$

(**3**) $\quad |\mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot P(\mathbf{y}) - \mathbf{y} \cdot P(\mathbf{x})| \leq \|\mathbf{x}\| \|\mathbf{y}\|$.

b)

By equality condition in inequality Cauchy in (**1**) equality occurs iff

$\mathbf{y} - 2P(\mathbf{y}) = k\mathbf{x}$, $\mathbf{x} - 2P(\mathbf{x}) = l\mathbf{y}$, where $k, l > 0$.

Identities $\|\mathbf{y} - 2P(\mathbf{y})\| = \|\mathbf{y}\|, \|\mathbf{x} - 2P(\mathbf{x})\| = \|\mathbf{x}\|$ imply $k = \dfrac{\|\mathbf{y}\|}{\|\mathbf{x}\|}$ and $l = \dfrac{\|\mathbf{x}\|}{\|\mathbf{y}\|}$.

Thus we have $\mathbf{y} - 2P(\mathbf{y}) = \dfrac{\mathbf{x}\|\mathbf{y}\|}{\|\mathbf{x}\|} \Leftrightarrow$

(4) $\mathbf{y}\|\mathbf{x}\| - \mathbf{x}\|\mathbf{y}\| = 2P(\mathbf{y})\|\mathbf{x}\|$

and $\mathbf{x} - 2P(\mathbf{x}) = \dfrac{\mathbf{y}\|\mathbf{x}\|}{\|\mathbf{y}\|} \Leftrightarrow$

(**5**) $\mathbf{x}\|\mathbf{y}\| - \mathbf{y}\|\mathbf{x}\| = 2P(\mathbf{x})\|\mathbf{y}\|.$

Adding and subtracting (**4**) and (**5**)we obtain:

$$0 = P(\mathbf{x})\|\mathbf{y}\| + P(\mathbf{y})\|\mathbf{x}\| = P(\mathbf{x}\|\mathbf{y}\| + \mathbf{y}\|\mathbf{x}\|) \Leftrightarrow$$

(**6**) $\mathbf{x}\|\mathbf{y}\| + \mathbf{y}\|\mathbf{x}\| \in \mathbf{F}^{\perp}$

and

$$P(\mathbf{x})\|\mathbf{y}\| - P(\mathbf{y})\|\mathbf{x}\| = \mathbf{x}\|\mathbf{y}\| - \mathbf{y}\|\mathbf{x}\| \Leftrightarrow P(\mathbf{x}\|\mathbf{y}\| - \mathbf{y}\|\mathbf{x}\|) = \mathbf{x}\|\mathbf{y}\| - \mathbf{y}\|\mathbf{x}\| \Leftrightarrow$$

(**7**) $\mathbf{x}\|\mathbf{y}\| - \mathbf{y}\|\mathbf{x}\| \in \mathbf{F}.$

It is mean there are $\mathbf{h} \in \mathbf{F}^{\perp}$ and $\mathbf{f} \in \mathbf{F}$ that

$$\frac{\mathbf{x}}{\|\mathbf{x}\|} + \frac{\mathbf{y}}{\|\mathbf{y}\|} = 2\mathbf{h} \text{ and } \frac{\mathbf{x}}{\|\mathbf{x}\|} - \frac{\mathbf{y}}{\|\mathbf{y}\|} = 2\mathbf{f} \Leftrightarrow \frac{\mathbf{x}}{\|\mathbf{x}\|} = \mathbf{h} + \mathbf{f} \text{ and } \frac{\mathbf{y}}{\|\mathbf{y}\|} = \mathbf{h} - \mathbf{f}.$$

Thus $\|\mathbf{h}\|^2 + \|\mathbf{f}\|^2 = 1$ and $\mathbf{x} = a(\mathbf{h} + \mathbf{f}), \mathbf{y} = b(\mathbf{h} - \mathbf{f}),$ where a, b be arbitrary positive real numbers, and we have

$$\mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot P(\mathbf{y}) - \mathbf{y} \cdot P(\mathbf{x}) = ab\left(\|\mathbf{h}\|^2 - \|\mathbf{f}\|^2\right) - ab(\mathbf{h} + \mathbf{f}) \cdot P(\mathbf{h} - \mathbf{f}) - ab(\mathbf{h} - \mathbf{f}) \cdot P(\mathbf{h} + \mathbf{f}) =$$
$$ab\left(\|\mathbf{h}\|^2 - \|\mathbf{f}\|^2 + \|\mathbf{f}\|^2 + \|\mathbf{f}\|^2\right) = ab\left(\|\mathbf{h}\|^2 + \|\mathbf{f}\|^2\right) = ab = \|\mathbf{x}\|\|\mathbf{y}\|.$$

Accordingly to substitution $\mathbf{y} := -\mathbf{y}$ which transform inequality (**1**) to inequality (**2**) we obtain

equality condition in inequality (**2**): $\mathbf{x} = a(\mathbf{h} + \mathbf{f}), \mathbf{y} = b(\mathbf{f} - \mathbf{h}),$ where a, b be arbitrary positive real numbers and $\mathbf{h} \in \mathbf{F}^{\perp}$, $\mathbf{f} \in \mathbf{F}$ with $\|\mathbf{h}\|^2 + \|\mathbf{f}\|^2 = 1.$

So, in inequality (**3**) equality occurs iff $\mathbf{x} = a(\mathbf{h} + \mathbf{f}), \mathbf{y} = b(\mathbf{h} - \mathbf{f}),$ where a, b be arbitrary real numbers and $\mathbf{h} \in \mathbf{F}^{\perp}$, $\mathbf{f} \in \mathbf{F}$ with $\|\mathbf{h}\|^2 + \|\mathbf{f}\|^2 = 1.$

Comment.

Since $\mathbf{x} \cdot P(\mathbf{y}) = \mathbf{y} \cdot P(\mathbf{x})$ inequality (**3**) can be rewritten in asymmetric form

$|\mathbf{x} \cdot \mathbf{y} - 2\mathbf{x} \cdot P(\mathbf{y})| \le \|\mathbf{x}\|\|\mathbf{y}\|$ or

$|\mathbf{x} \cdot \mathbf{y} - 2\mathbf{y} \cdot P(\mathbf{x})| \le \|\mathbf{x}\|\|\mathbf{y}\|.$

Remark.

The problem represented above is generalization of

Problem.(**#1193**,**Kvant N11**, **1989**,**Vasile Cirtoaje** / **Old and new p.18**)

For any real number a, b, c, x, y, z prove inequality

$$ax + by + cz + \sqrt{(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)} \ge \frac{2}{3}(a + b + c)(x + y + z).$$