W58. Fibonacci numbers and divisibility 2.

Problem with a solution proposed by Arkady Alt, San Jose , California, USA.

Let f_n be $n-th$ Fibonacci number defined by recurrence $f_{n+1}-f_n-f_{n-1}=0, n\in\mathbb{N}$ and initial conditions $f_0=0, f_1=1.$

Prove that $(5n^2+3n-2)f_n-6nf_{n+1}$ is divisible by 50 for any $n\in\mathbb{N}$

Solution.

Let $h_n:=\dfrac{(5n^2+3n-2)f_n-6nf_{n+1}}{50}, n\in\mathbb{N}\cup\{0\}$. Then $h_0=0, h_1=0.$

Note that $h_{n+1}=\dfrac{\left(5(n+1)^2+3(n+1)-2\right)f_{n+1}-6(n+1)f_{n+2}}{50}=$
$\dfrac{(5n^2+13n+6)f_{n+1}-6(n+1)(f_{n+1}+f_n)}{50}=\dfrac{n(5n+7)f_{n+1}-6(n+1)f_n}{50}.$

Then

$$h_{n+2}=\frac{(n+1)(5(n+1)+7)f_{n+2}-6(n+2)f_{n+1}}{50}=\frac{(5n^2+17n+12)(f_{n+1}+f_n)-6(n+2)f_{n+1}}{50}=$$

$\dfrac{(5n^2+17n+12)f_n+n(5n+11)f_{n+1}}{50}$ and, therefore,

$$h_{n+2}-h_{n+1}-h_n=\frac{(5n^2+17n+12)f_n+n(5n+11)f_{n+1}}{50}-\frac{n(5n+7)f_{n+1}-6(n+1)f_n}{50}-$$
$$\frac{(5n^2+3n-2)f_n-6nf_{n+1}}{50}=\frac{nf_{n+1}+2(n+1)f_n}{5}.$$

Let $g_n:=\dfrac{nf_{n+1}+2(n+1)f_n}{5},\ n\in\mathbb{N}\cup\{0\}$. Then $g_0=0, g_1=\dfrac{f_2+4f_1}{5}=1$ and we have $g_{n+1}=\dfrac{(n+1)f_{n+2}+2(n+2)f_{n+1}}{5}=\dfrac{(n+1)(f_{n+1}+f_n)+2(n+2)f_{n+1}}{5}=$
$\dfrac{(n+1)f_n+(3n+5)f_{n+1}}{5}$. Also, $g_{n-1}=\dfrac{(n-1)f_n+2nf_{n-1}}{5}=\dfrac{(n-1)f_n+2n(f_{n+1}-f_n)}{5}=$
$\dfrac{2nf_{n+1}-(n+1)f_n}{5}.$

Hence,

$$g_{n+1}-g_n-g_{n-1}=\frac{(n+1)f_n+(3n+5)f_{n+1}}{5}-\frac{nf_{n+1}+2(n+1)f_n}{5}-\frac{2nf_{n+1}-(n+1)f_n}{5}=f_{n+1}.$$

Since g_n is integer for any $n\in\mathbb{N}\cup\{0\}$ ($g_0=0, g_1=1$ and $g_{n+1}=g_n+g_{n-1}+f_{n+1}, n\in\mathbb{N}$) and $h_{n+2}=h_{n+1}+h_n+g_n, n\in\mathbb{N}\cup\{0\}$, where $h_0=h_1=0$ then for any $n\in\mathbb{N}\cup\{0\}$ h_n is integer as well.