W53. If $a,b,c>0$ then

$$\prod_{k=1}^{n}\frac{a^{2k}+b^{2k}+c^{2k}}{a^k+b^k+c^k}\geq\left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{\frac{n(n+1)}{2}}.$$

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Let $A_n:=\dfrac{a^{n+1}+b^{n+1}+c^{n+1}}{a^n+b^n+c^n}, n\in\mathbb{N}\cup\{0\}, a,b,c>0.$ We will prove that for any $n\in\mathbb{N}\cup\{0\}$ holds inequality $A_{n+1}\geq A_n$.

Indeed, $\dfrac{a^{n+2}+b^{n+2}+c^{n+2}}{a^{n+1}+b^{n+1}+c^{n+1}}\geq\dfrac{a^{n+1}+b^{n+1}+c^{n+1}}{a^n+b^n+c^n}\Leftrightarrow\sum a^{n+2}\cdot\sum a^n\geq\left(\sum a^{n+1}\right)^2\Leftrightarrow$

$\sum a^{2n+2}+\sum a^n(b^{n+2}+c^{n+2})\geq\sum a^{2n+2}+2\sum b^{n+1}c^{n+1}\Leftrightarrow$

$\sum(a^nb^{n+2}+a^nb^{n+2})\geq 2\sum a^{n+1}b^{n+1}\Leftrightarrow\sum a^nb^n(a-b)^2\geq 0.$

Thus, for any $n\in\mathbb{N}$ holds $A_n\geq A_1$. Since $\dfrac{a^{2n}+b^{2k}+c^{2k}}{a^n+b^n+c^n}=\prod_{k=n}^{2n-1}A_k\geq A_1^n$

then $\displaystyle\prod_{k=1}^{n}\frac{a^{2k}+b^{2k}+c^{2k}}{a^k+b^k+c^k}\geq\prod_{k=1}^{n}A_1^k\geq A_1^{1+2+\ldots+n}=A_1^{\frac{n(n+1)}{2}}=\left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{\frac{n(n+1)}{2}}.$