W41. Let $(x_n)_{n\geq 1},\ (y_n)_{n\geq 1}$ be two sequences of positive real numbers such that

$$\lim_{n\to\infty}\frac{y_n}{n} = 0,\ \lim_{n\to\infty}\frac{y_n^2}{n} = \beta \in \mathbb{R}, \lim_{n\to\infty} x_n = 0,\ \ \lim_{n\to\infty}\left(\sum_{k=1}^{n} x_k - y_n\right) = \alpha \in \mathbb{R}$$

Then $\lim_{n\to\infty}\left(n\left(\left(1+\frac{x_1}{n}\right)\left(1+\frac{x_1}{n}\right)\ldots\left(1+\frac{x_n}{n}\right)-1\right)-y_n\right) = \alpha + \frac{1}{2}\beta.$

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First we will analyze properties of the sequences $(x_n)_{n\geq 1},\ (y_n)_{n\geq 1}$ given in the statement of the problem.

1. Let $\beta_n := \frac{y_n^2}{n} - \beta, n \in \mathbb{N}$. Then $\lim_{n\to\infty}\frac{y_n^2}{n} = \beta \Leftrightarrow \frac{y_n^2}{n} = \beta + \beta_n \Leftrightarrow y_n = \sqrt{n(\beta+\beta_n)}$,

where $\lim_{n\to\infty}\beta_n = 0$ and, therefore, $\lim_{n\to\infty}\frac{y_n}{n} = \lim_{n\to\infty}\frac{\sqrt{n(\beta+\beta_n)}}{n} = \lim_{n\to\infty}\sqrt{\frac{\beta+\beta_n}{n}} = 0$

implies $\lim_{n\to\infty}\frac{y_n}{n} = 0$ (that is condition $\lim_{n\to\infty}\frac{y_n}{n} = 0$ is unnecessary)

and also $\lim_{n\to\infty}\frac{y_n}{\sqrt{n}} = \sqrt{\beta}$.

2. Let $\alpha_n := \sum_{k=1}^{n} x_k - y_n - \alpha, n \in \mathbb{N}$. Then $\lim_{n\to\infty}\left(\sum_{k=1}^{n} x_k - y_n\right) = \alpha \Leftrightarrow$

$\sum_{k=1}^{n} x_k = y_n + \alpha + \alpha_n$, where $\lim_{n\to\infty}\alpha_n = 0$ and since $x_n = y_n - y_{n-1} + \delta_n$, where

$\delta_n := \alpha_n - \alpha_{n-1}$ and $\lim_{n\to\infty}\delta_n = 0$. Hence, $x_n^2 = (y_n - y_{n-1})^2 + 2(y_n - y_{n-1})\delta_n + \delta_n^2$

and $\lim_{n\to\infty}\frac{x_n^2}{n} = \lim_{n\to\infty}\frac{(y_n - y_{n-1})^2 + 2(y_n - y_{n-1})\delta_n + \delta_n^2}{n} = \lim_{n\to\infty}\frac{(y_n - y_{n-1})^2}{n} =$

$\lim_{n\to\infty}\frac{y_n^2}{n} + \lim_{n\to\infty}\frac{y_{n-1}^2}{n} - 2\lim_{n\to\infty}\frac{y_n y_{n-1}}{n} = 2\beta - 2\lim_{n\to\infty}\frac{y_n}{\sqrt{n}}\cdot\frac{y_{n-1}}{\sqrt{n-1}}\cdot\frac{\sqrt{n-1}}{\sqrt{n}} =$

$2\beta - 2\left(\sqrt{\beta}\right)^2 = 0.$

3. Since $\lim_{n\to\infty} x_n = 0 \Rightarrow \lim_{n\to\infty} x_n^2 = 0$ then by Stoltz-Cesaro Theorem $\lim_{n\to\infty}\frac{\sum_{k=1}^{n} x_n^2}{n} = 0$

and, therefore, $\lim_{n\to\infty}\frac{1}{n}\sum_{1\leq i<j\leq n} x_i x_j = \lim_{n\to\infty}\frac{1}{2n}\left(\left(\sum_{k=1}^{n} x_k\right)^2 - \sum_{k=1}^{n} x_k^2\right) = \lim_{n\to\infty}\frac{1}{2n}\left(\sum_{k=1}^{n} x_k\right)^2 =$

$\frac{1}{2}\lim_{n\to\infty}\frac{1}{n}(y_n + \alpha + \alpha_n)^2 = \frac{1}{2}\lim_{n\to\infty}\frac{y_n^2 + 2y_n(\alpha+\alpha_n) + (\alpha+\alpha_n)^2}{n} = \frac{1}{2}\lim_{n\to\infty}\frac{y_n^2}{n} = \frac{\beta}{2}.$

4. Let $P_n := \prod_{k=1}^{n}\left(1+\frac{x_k}{n}\right), S_n := \sum_{k=1}^{n} x_k$ and let

$\sigma_k = \sigma_k(x_1, x_2, \ldots, x_n) := \sum_{1\leq i_1<i_2<\ldots<i_k\leq n} x_{i_1}x_{i_2}\ldots x_{i_k}$

for any positive real numbers $x_1, x_2, \ldots, x_n$, where $k = 1, 2, \ldots, n$.

We have $P_n = 1 + \frac{S_n}{n} + \frac{1}{n^2}\sum_{1\leq i<j\leq n} x_i x_j + R_n$, where $R_n := \sum_{k=3}^{n}\frac{1}{n^k}\sigma_k$.

Applying Maclaurin's inequality $\sqrt[k]{\frac{\sigma_k}{\binom{n}{k}}} \leq \frac{\sigma_1}{n}$ in the form $\sigma_k \leq \frac{\binom{n}{k}}{n^k}\sigma_1$

we obtain $\frac{1}{\sqrt{i_1 \cdot i_2 \cdot \ldots \cdot i_k}} \leq \frac{\binom{n}{k}}{n^k}\sigma_1^k = \frac{\binom{n}{k}}{n^k}S_n^k$ and, therefore,

$$R_n \leq \sum_{k=3}^{n} \frac{1}{n^k} \cdot \frac{\binom{n}{k}}{n^k}S_n^k.$$

Noting that $\frac{\binom{n}{k}}{n^k} < \frac{1}{k!}$ and $S_n = y_n + \alpha + \alpha_n$ we obtain

$$R_n \leq \sum_{k=3}^{n} \frac{1}{n^k k!}(y_n + \alpha + \alpha_n)^k = \sum_{k=3}^{n} \frac{1}{n^{k/2}} \frac{1}{k!}\left(\frac{y_n + \alpha + \alpha_n}{\sqrt{n}}\right)^k.$$

Since $\lim\limits_{n\to\infty} \frac{y_n}{\sqrt{n}} = \sqrt{\beta}$ then there is some $M > 0$ such that $\frac{y_n + \alpha + \alpha_n}{\sqrt{n}} < M$

and, therefore, $R_n < \sum_{k=3}^{n} \frac{1}{n^{k/2}} \frac{M^k}{k!} < \sum_{k=3}^{n} \frac{1}{n^{3/2}} \frac{M^k}{k!} < \frac{e^M}{n^{3/2}}$

Thus, $0 < P_n - 1 - \frac{S_n}{n} - \frac{1}{n^2} \sum_{1 \leq i < j \leq n} x_i x_j = R_n < \frac{e^M}{n^{3/2}} \Leftrightarrow$

$0 < n(P_n - 1) - S_n - \frac{1}{n} \sum_{1 \leq i < j \leq n} x_i x_j < \frac{e^M}{n^{1/2}}$ and, therefore,

$$\lim_{n\to\infty}\left(n(P_n - 1) - S_n - \frac{1}{n} \sum_{1 \leq i < j \leq n} x_i x_j\right) = 0 \Leftrightarrow$$

$$\lim_{n\to\infty}\left(n(P_n - 1) - y_n - \alpha - \frac{1}{n} \sum_{1 \leq i < j \leq n} x_i x_j\right) = 0 \Leftrightarrow.$$

$$\lim_{n\to\infty}(n(P_n - 1) - y_n) = \alpha + \lim_{n\to\infty} \frac{1}{n} \sum_{1 \leq i < j \leq n} x_i x_j = \alpha + \frac{\beta}{2}.$$