**Fibonacci numbers and divisibility.**

**Problem with a solution proposed by Arkady Alt, San Jose , California, USA.**

Let $f_n$ be $n-th$ Fibonacci number defined by recurrence $f_{n+1}-f_n-f_{n-1}=0, n\in\mathbb{N}$ and initial conditions $f_0=0, f_1=1.$ Prove that for any $n\in\mathbb{N}$ $(n-1)(n+1)(2nf_{n+1}-(n+6)f_n)$ is divisible by $150$ for any $n\in\mathbb{N}$.

**Solution.**

Let $s_n:=\dfrac{(n-1)(n+1)(2nf_{n+1}-(n+6)f_n)}{150}, n\in\mathbb{N}\cup\{0\}$. Then $s_0=0, s_1=0.$

Note that $s_{n+1}=\dfrac{n(n+2)(2(n+1)f_{n+2}-(n+7)f_{n+1})}{150}=$ $\dfrac{n(n+2)(2(n+1)(f_{n+1}+f_n)-(n+7)f_{n+1})}{150}=\dfrac{n(n+2)((n-5)f_{n+1}+2(n+1)f_n)}{150}$

and $s_{n-1}=\dfrac{(n-2)n(2(n-1)f_n-(n+5)f_{n-1})}{150}=\dfrac{(n-2)n(2(n-1)f_n-(n+5)(f_{n+1}-f_n))}{150}=$ $\dfrac{(n-2)n(3(n+1)f_n-(n+5)f_{n+1})}{150}$. Hence,

$s_{n+1}-s_n-s_{n-1}=\dfrac{n(n+2)((n-5)f_{n+1}+2(n+1)f_n)}{150}-\dfrac{(n-1)(n+1)(2nf_{n+1}-(n+6)f_n)}{150}-$ $\dfrac{(n-2)n(3(n+1)f_n-(n+5)f_{n+1})}{150}=\dfrac{(5n^2+3n-2)f_n-6nf_{n+1}}{50}.$

Let $h_n:=\dfrac{(5n^2+3n-2)f_n-6nf_{n+1}}{50}, n\in\mathbb{N}\cup\{0\}$. Then $h_0=0, h_1=0.$

Noting that $h_{n+1}=\dfrac{\left(5(n+1)^2+3(n+1)-2\right)f_{n+1}-6(n+1)f_{n+2}}{50}=$ $\dfrac{(5n^2+13n+6)f_{n+1}-6(n+1)(f_{n+1}+f_n)}{50}=\dfrac{n(5n+7)f_{n+1}-6(n+1)f_n}{50}$ we obtain

$h_{n+2}=\dfrac{(n+1)(5(n+1)+7)f_{n+2}-6(n+2)f_{n+1}}{50}=\dfrac{(5n^2+17n+12)(f_{n+1}+f_n)-6(n+2)f_{n+1}}{50}=$ $\dfrac{(5n^2+17n+12)f_n+n(5n+11)f_{n+1}}{50}$ and, therefore,

$h_{n+2}-h_{n+1}-h_n=\dfrac{(5n^2+17n+12)f_n+n(5n+11)f_{n+1}}{50}-\dfrac{n(5n+7)f_{n+1}-6(n+1)f_n}{50}-$ $\dfrac{(5n^2+3n-2)f_n-6nf_{n+1}}{50}=\dfrac{nf_{n+1}+2(n+1)f_n}{5}.$

Let $g_n:=\dfrac{nf_{n+1}+2(n+1)f_n}{5},\ n\in\mathbb{N}\cup\{0\}$. Then $g_0=0, g_1=\dfrac{f_2+4f_1}{5}=1$ and we have $g_{n+1}=\dfrac{(n+1)f_{n+2}+2(n+2)f_{n+1}}{5}=\dfrac{(n+1)(f_{n+1}+f_n)+2(n+2)f_{n+1}}{5}=$ $\dfrac{(n+1)f_n+(3n+5)f_{n+1}}{5}$. Also, $g_{n-1}=\dfrac{(n-1)f_n+2nf_{n-1}}{5}=\dfrac{(n-1)f_n+2n(f_{n+1}-f_n)}{5}=$ $\dfrac{2nf_{n+1}-(n+1)f_n}{5}$.

Hence,

$$g_{n+1}-g_n-g_{n-1}=\frac{(n+1)f_n+(3n+5)f_{n+1}}{5}-\frac{nf_{n+1}+2(n+1)f_n}{5}-\frac{2nf_{n+1}-(n+1)f_n}{5}=f_{n+1}.$$

Since $g_n$ is integer for any $n\in\mathbb{N}\cup\{0\}$ (By Math Induction because $g_0=0, g_1=1$ and $g_{n+1}=g_n+g_{n-1}+f_{n+1}, n\in\mathbb{N}$)
and $h_{n+2}=h_{n+1}+h_n+g_n, n\in\mathbb{N}\cup\{0\}$, where $h_0=h_1=0$ then for any $n\in\mathbb{N}\cup\{0\}$ $h_n$ is integer as well. Thus, $s_{n+1}=s_n+s_{n-1}+h_n, n\in\mathbb{N}\cup\{0\}$, where $s_0=0, s_1=0$

and $h_n$ is integer for any $n \in \mathbb{N} \cup \{0\}$. Hence, again by Math Induction, we can conclude that

$s_n$ is integer for any $n \in \mathbb{N} \cup \{0\}$.