W39. Let $a,b,c \in \mathbb{R}$. Solve the equations:

$$x^3 - 3ax^2 + 3(a^2 - b^2)x - a^3 + 3ab^2 - 2b^3 = 0 \quad \textbf{(1)}$$

$$x^3 - 3ax^2 + (3a^2 - b^2 - c^2 - bc)x - a^3 + ab^2 + ac^2 - b^2c - bc^2 + abc = 0 \quad \textbf{(2)}$$

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1. Since $x^3 - 3ax^2 + 3(a^2 - b^2)x - a^3 + 3ab^2 - 2b^3 = -(a+2b-x)(a-b-x)^2 =$
$x^3 - 3x^2a + 3xa^2 - a^3 - 3b^2x + 3b^2a - 2b^3 = (x-a)^3 - 3b^2(x-a) - 2b^3 =$
$t^3 - 3b^2t - 2b^3$,where $t := x - a$ and $t^3 - 3b^2t - 2b^3 = t^3 - b^2t - 2b^2t - 2b^3 =: -$
$(t+b)(t^2 - bt) - 2b^2(t+b) = (t+b)(t^2 - bt - 2b^2) = (t+b)(t+b)(t-2b) =$
$(t+b)^2(t-2b)$ then $(x-a)^3 - 3b^2(x-a) - 2b^3 = (x-a-2b)(x-a+b)^2$
and, therefore,$x^3 - 3ax^2 + 3(a^2 - b^2)x - a^3 + 3ab^2 - 2b^3 = 0 \Leftrightarrow$

$$(x-a-2b)(x-a+b)^2 = 0 \Leftrightarrow \left[\begin{array}{l} x = a+2b \\ x = a-b \end{array}\right. .$$

2. $x^3 - 3ax^2 + (3a^2 - b^2 - c^2 - bc)x - a^3 + ab^2 + ac^2 - b^2c - bc^2 + abc =$
$(x-a)^3 - b^2(x-a) - c^2(x-a) - bc(x-a) - bc(b+c) =$
$t^3 - t(b^2 + bc + c^2) - bc(b+c)$,where $t := x - a$ and
$t^3 - t(b^2 + bc + c^2) - bc(b+c) = t^3 - t(b+c)^2 + bc(t - (b+c)) =$
$(t-b-c)(t^2 - (b+c)t + bc) = (t-b-c)(t-b)(t-c)$ then
$x^3 - 3ax^2 + (3a^2 - b^2 - c^2 - bc)x - a^3 + ab^2 + ac^2 - b^2c - bc^2 + abc = 0 \Leftrightarrow$

$$(x-a-b-c)(x-a-b)(x-a-c) = 0 \Leftrightarrow \left[\begin{array}{l} x = a+b+c \\ x = a+b \\ x = a+c \end{array}\right.$$