W39. Find the following limits:

a). $\lim\limits_{n\to\infty}\dfrac{\left(1+\dfrac{1}{n(n+a)}\right)^{n^3}}{\left(1+\dfrac{1}{n+b}\right)^{n^2}}$.

b). $\lim\limits_{n\to\infty}(n+1)\left(\left(1+\dfrac{1}{n(n+1)}\right)^{(n+2)n}-\left(1+\frac{1}{n}\right)^n\right)$.

c). $\lim\limits_{n\to\infty}\left((n+2)\left(1+\dfrac{1}{n(n+1)}\right)^{(n+1)n}-(n+1)\left(1+\frac{1}{n}\right)^n\right)$.

Solution by the proposer.

a) $\ln\dfrac{\left(1+\dfrac{1}{n(n+a)}\right)^{n^3}}{\left(1+\dfrac{1}{n+b}\right)^{n^2}}=n^3\ln\left(1+\dfrac{1}{n(n+a)}\right)-n^2\ln\left(1+\dfrac{1}{n+b}\right)=$

$n^3\left(\dfrac{1}{n(n+a)}-\dfrac{1}{n^2(n+a)^2}+o\left(\dfrac{1}{n^4}\right)\right)-n^2\left(\dfrac{1}{n+b}-\dfrac{1}{2(n+b)^2}+o\left(\dfrac{1}{n^2}\right)\right)=$

$\dfrac{n^2}{n+a}-\dfrac{n}{(n+a)^2}-\dfrac{n^2}{n+b}+\dfrac{n^2}{2(n+b)^2}+o(1)=$

$\dfrac{n^2(b-a)}{(n+a)(n+b)}+\dfrac{n^2}{2(n+b)^2}+o(1)\sim b-a+\dfrac{1}{2}$.

Hence, $\lim\limits_{n\to\infty}\dfrac{\left(1+\dfrac{1}{n(n+a)}\right)^{n^3}}{\left(1+\dfrac{1}{n+b}\right)^{n^2}}=e^{b-a}\sqrt{e}$.

b) Let $a_n=\left(1+\dfrac{1}{n(n+1)}\right)^{(n+2)n}$, $e_n=\left(1+\frac{1}{n}\right)^n$. Since $\frac{a_n}{e_n}\sim 1$ and

$\ln(1+t)=t-\dfrac{t^2}{2}+o(t)$ we have

$(n+1)\left(\left(1+\dfrac{1}{n(n+1)}\right)^{(n+2)n}-\left(1+\frac{1}{n}\right)^n\right)\sim en\left(\frac{a_n}{e_n}-1\right)\sim en\ln\left(\frac{a_n}{e_n}\right)\sim$

$e\left(n^2(n+2)\ln\left(1+\dfrac{1}{n(n+1)}\right)-n^2\ln\left(1+\frac{1}{n}\right)\right)=e\left(\dfrac{n(n+2)}{n+1}-n+\frac{1}{2}+o(1)\right)=$

$e\left(\dfrac{n}{n+1}+\frac{1}{2}+o(1)\right)\sim\dfrac{3e}{2}$.

Hence, $\lim\limits_{n\to\infty}(n+1)\left(\left(1+\dfrac{1}{n(n+1)}\right)^{(n+2)n}-\left(1+\frac{1}{n}\right)^n\right)=\dfrac{3e}{2}$.

c) Let $e_n=\left(1+\frac{1}{n}\right)^n$, then $(n+2)\left(1+\dfrac{1}{n(n+1)}\right)^{(n+1)n}-(n+1)\left(1+\frac{1}{n}\right)^n=$

$(n+2)e_{n(n+1)}-(n+1)e_n$. Since, $e_n\sim e$ and $\dfrac{e_{n(n+1)}}{e_n}\sim 1$ then $(n+2)e_{n(n+1)}-(n+1)e_n=$

$(n+1)e_n\left(\dfrac{(n+2)e_{n(n+1)}}{(n+1)e_n}-1\right)\sim en\ln\left(\dfrac{(n+2)e_{n(n+1)}}{(n+1)e_n}\right)=e\left(n\ln\left(\dfrac{n+2}{n+1}\right)+n(\ln e_{n(n+1)}-\ln e_n)\right.$

We have $n(\ln e_{n(n+1)}-\ln e_n)=(n+1)n^2\ln\left(1+\dfrac{1}{n(n+1)}\right)-n^2\ln\left(1+\frac{1}{n}\right)=$

$$(n+1)n^2\left(\frac{1}{n(n+1)}-\frac{1}{2n^2(n+1)^2}+o\left(\frac{1}{n^4}\right)\right)-n^2\left(\frac{1}{n}-\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right)=\frac{1}{2}+o(1).$$

Hence, $\lim\limits_{n\to\infty}((n+2)e_{n(n+1)}-(n+1)e_n)=e\lim\limits_{n\to\infty}\left(n\ln\left(\frac{n+2}{n+1}\right)+n(\ln e_{n(n+1)}-\ln e_n)\right)=$
$e\left(\lim\limits_{n\to\infty}n\ln\left(\frac{n+2}{n+1}\right)+\lim\limits_{n\to\infty}n(\ln e_{n(n+1)}-\ln e_n)\right)=e\left(1+\frac{1}{2}\right)=\frac{3e}{2}.$

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