One inequality for area and squares of sidelengths in an acute triangle.
Problem with a solution proposed by Arkady Alt, San Jose, California, USA.

Let a,b,c *be the sides of an acute triangle* ΔABC , *then for any* $x,y,z \geq 0,$ *such that* $xy + yz + zx = 1,$ *holds inequality:*

$$a^2x + b^2y + c^2z \geq 4F, \text{ where } F \text{ is the area of the triangle } \Delta ABC.$$

Solution.

Using interpretation x,y,z as the $\cot\alpha, \cot\beta, \cot\gamma$ correspondingly, for $\alpha + \beta + \gamma = \pi$ and $\alpha,\beta,\gamma \in \left(0, \frac{\pi}{2}\right]$ $(x = \cot\alpha, y = \cot\beta, z = \cot\gamma)$ we can rewrite inequality as

$$a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma \geq 4F = a^2\cot\alpha_* + b^2\cot\beta_* + c^2\cot\gamma_*$$

At the first stage we will prove intermediate inequality for any $\alpha,\beta \in \left(0, \frac{\pi}{2}\right]$

(**1**) $a^2\cot\alpha + b^2\cot\beta \geq \dfrac{2ab}{\sin(\alpha+\beta)} + (a^2+b^2)\cot(\alpha+\beta) \Leftrightarrow a^2(\cot\alpha - \cot(\alpha+\beta)) + b^2(\cot\beta - \cot(\alpha+\beta)) \geq \dfrac{2ab}{\sin(\alpha+\beta)}$. But latter inequality is valid, because

$$a^2(\cot\alpha - \cot(\alpha+\beta)) + b^2(\cot\beta - \cot(\alpha+\beta)) = \frac{a^2\sin\beta}{\sin\alpha\sin(\alpha+\beta)} + \frac{b^2\sin\alpha}{\sin\beta\sin(\alpha+\beta)} \geq \frac{2ab}{\sin(\alpha+\beta)}.$$

Equality occurs when $\dfrac{a^2\sin\beta}{\sin\alpha\sin(\alpha+\beta)} = \dfrac{b^2\sin\alpha}{\sin\beta\sin(\alpha+\beta)} \Leftrightarrow \dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$.

Using inequality (**1**) we obtain inequality

$$a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma \geq \frac{2ab}{\sin\gamma} - (a^2+b^2)\cot\gamma + c^2\cot\gamma = \frac{2ab}{\sin\gamma} - (a^2+b^2-c^2)\cot\gamma.$$

Now we will go on to find the minimum of function $h(\gamma) := \dfrac{2ab}{\sin\gamma} - (a^2+b^2-c^2)\cot\gamma$ on interval $\left(0, \frac{\pi}{2}\right]$.

$h'(\gamma) = \dfrac{-2ab\cos\gamma + (a^2+b^2-c^2)}{\sin^2\gamma} = \dfrac{2ab(\cos\gamma_* - \cos\gamma)}{\sin^2\gamma}$. So, $h(\gamma)$ decrease on $(0,\gamma_*]$ and increases on $\left[\gamma_*, \frac{\pi}{2}\right]$. Thus, $\min h(\gamma) = h(\gamma_*)$, and this local minimum at the same time is a global minimum. So we obtain a lower bound for $a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma$,which is $\dfrac{2ab}{\sin\gamma_*} - (a^2+b^2-c^2)\cot\gamma_*$,

and it can be reached when:

$\gamma = \gamma_*$ and $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} \Leftrightarrow \alpha = \alpha_*, \beta = \beta_*, \gamma = \gamma_*$.

Really, $\cot\gamma_* = \dfrac{a^2+b^2-c^2}{4K}$, $\cot\alpha_* = \dfrac{b^2+c^2-a^2}{4K}$ and $\dfrac{b}{a\sin\gamma} = \dfrac{b}{a\sin\gamma_*} = \dfrac{b^2}{ab\sin\gamma_*} = \dfrac{b^2}{2K}$, and we have $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} \Leftrightarrow \dfrac{a}{\sin\alpha} = \dfrac{b}{\sin(\gamma+\alpha)} \Leftrightarrow$

$a\sin(\gamma+\alpha) = b\sin\alpha \Leftrightarrow a\sin\gamma\cos\alpha + a\sin\alpha\cos\gamma - b\sin\alpha = 0 \Leftrightarrow$

$$\cot\alpha = \frac{b}{a\sin\gamma} - \cot\gamma = \frac{b}{a\sin\gamma_*} - \cot\gamma_* = \frac{b^2}{2K} - \frac{a^2+b^2-c^2}{4K} = \frac{b^2+c^2-a^2}{4K} = \cot\alpha_*.$$

In the same manner we obtain $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \Leftrightarrow \cot\beta = \frac{a}{b\sin\gamma} - \cot\gamma =$
$\frac{a^2}{2K} - \frac{a^2+b^2-c^2}{4K} = \frac{a^2+c^2-b^2}{4K} = \cot\beta_*$.
Thus, $a^2\cot\alpha_* + b^2\cot\beta_* + c^2\cot\gamma_* = h(\gamma_*) \leq a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma$
and because $\frac{a}{2}\cot\alpha_*, \frac{b}{2}\cot\beta_*, \frac{c}{2}\cot\gamma_*$are the distances from circumcenter
to the sides a,b,c correspondingly, then $a^2\cot\alpha_* = 4F_a,\ b^2\cot\beta_* = 4F_b, c^2\cot\gamma_* = 4F_c$
and $a^2\cot\alpha_* + b^2\cot\beta_* + c^2\cot\gamma_* = 4(F_a+F_b+F_c) = 4F.$
(F_a,F_b,F_c are areas of triangles $\Delta BOC,\Delta COA,\Delta AOB$ correspondingly).