Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Let $\Delta^3(\sqrt{n}) := \sqrt{n+3} - 3\sqrt{n+2} + 3\sqrt{n+1} - \sqrt{n}$. Find $\lim\limits_{n\to\infty} n^{5/2}\Delta^3(\sqrt{n})$.

Solution1.

Let $\delta_n := \Delta^2(\sqrt{n}) + \dfrac{1}{4\sqrt{n(n+1)(n+2)}}$. Since $\Delta^2(\sqrt{n}) = \sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n} =$

$$\frac{1}{\sqrt{n+2}+\sqrt{n+1}} - \frac{1}{\sqrt{n+1}+\sqrt{n}} = -\frac{\sqrt{n+2}-\sqrt{n}}{\left(\sqrt{n+2}+\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n}\right)} =$$

$-\dfrac{2}{\left(\sqrt{n+2}+\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n+2}+\sqrt{n}\right)}$ then

$$\delta_{n-1} = -\frac{2}{\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n}+\sqrt{n-1}\right)\left(\sqrt{n-1}+\sqrt{n+1}\right)} + \frac{1}{4\sqrt{(n-1)n(n+1)}}.$$

Noting that $\sqrt{n+1} + \sqrt{n} \ge 2\sqrt[4]{n(n+1)}$, $\sqrt{n} + \sqrt{n-1} \ge 2\sqrt[4]{n(n-1)}$ and $\sqrt{n-1} + \sqrt{n+1} \ge 2\sqrt[4]{(n-1)(n+1)}$ we obtain

$$\delta_{n-1} \ge -\frac{2}{2\sqrt[4]{n(n+1)}\cdot 2\sqrt[4]{n(n+1)}\cdot 2\sqrt[4]{(n-1)(n+1)}} + \frac{1}{4\sqrt{(n-1)n(n+1)}} = 0.$$

Also, using inequality $\sqrt{a} + \sqrt{b} \le \sqrt{2(a+b)}$ we obtain

$\sqrt{n+1} + \sqrt{n} \le \sqrt{2(n+1+n)} = \sqrt{2(2n+1)}$, $\sqrt{n} + \sqrt{n-1} \le \sqrt{2(2n-1)}$,

$\sqrt{n-1} + \sqrt{n+1} \le \sqrt{2\cdot 2n} = 2\sqrt{n}$ and, therefore,

$$\delta_{n-1} \le -\frac{2}{\sqrt{2(2n+1)}\cdot\sqrt{2(2n-1)}\cdot 2\sqrt{n}} + \frac{1}{4\sqrt{(n-1)n(n+1)}} =$$

$$-\frac{1}{4\sqrt{n^2-1/4}\cdot\sqrt{n}} + \frac{1}{4\sqrt{(n^2-1)n}} = \frac{1}{4\sqrt{n}}\left(\frac{1}{\sqrt{n^2-1}} - \frac{1}{\sqrt{n^2-1/4}}\right) =$$

$$\frac{3}{16\sqrt{n}\cdot\sqrt{n^2-1}\cdot\sqrt{n^2-1/4}\left(\sqrt{n^2-1}+\sqrt{n^2-1/4}\right)}.$$

Since $0 \le \delta_{n-1}\cdot n^{5/2} \le \dfrac{3n^{5/2}}{16\sqrt{n}\cdot\sqrt{n^2-1}\cdot\sqrt{n^2-1/4}\left(\sqrt{n^2-1}+\sqrt{n^2-1/4}\right)}$

then $\lim\limits_{n\to\infty}\delta_{n-1}\cdot n^{5/2} = 0 \iff \lim\limits_{n\to\infty}\delta_n\cdot n^{5/2} = 0 \iff \lim\limits_{n\to\infty}\delta_{n+1}\cdot n^{5/2} = 0.$

Using the latter limit we can find asymptotic behavior of $\Delta^3\sqrt{n}$, namely we will find $\lim\limits_{n\to\infty} n^{5/2}\Delta^3(\sqrt{n})$.

We have $\lim\limits_{n\to\infty} n^{5/2}\Delta^3(\sqrt{n}) = \lim\limits_{n\to\infty} n^{5/2}\left(\Delta^2\sqrt{n+1} - \Delta^2\sqrt{n}\right) =$

$$\lim_{n\to\infty} n^{5/2}\left(\left(\Delta^2\sqrt{n+1} + \frac{1}{4\sqrt{(n+1)(n+2)(n+3)}}\right) - \left(\Delta^2\sqrt{n} + \frac{1}{4\sqrt{n(n+1)(n+2)}}\right)\right) +$$

$$\lim_{n\to\infty} n^{5/2}\left(\frac{1}{4\sqrt{n(n+1)(n+2)}} - \frac{1}{4\sqrt{(n+1)(n+2)(n+3)}}\right) = \lim_{n\to\infty}(\delta_{n+1}\cdot n^{5/2} - \delta_n\cdot n^{5/2}) +$$

$$\lim_{n\to\infty}\frac{n^{5/2}}{4\sqrt{n(n+1)(n+2)(n+3)}}\left(\sqrt{n+3}-\sqrt{n}\right) = \frac{1}{4}\lim_{n\to\infty} n^{1/2}\left(\sqrt{n+3}-\sqrt{n}\right) = \frac{3}{8}.$$

Thus, $\quad \Delta^3(\sqrt{n}) \approx \dfrac{3}{8n^{5/2}} \approx \dfrac{3}{8\sqrt{n(n+1)(n+2)(n+3)(n+4)}}$.

Solution 2. (**direct way to find** $\lim\limits_{n\to\infty} n^{5/2}\Delta^3(\sqrt{n})$).

We have $\Delta^3\sqrt{n} = \Delta^2\sqrt{n+1} - \Delta^2\sqrt{n} =$

$\sqrt{n+3} - 2\sqrt{n+2} + \sqrt{n+1} - \left(\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}\right) =$

$-\dfrac{2}{\left(\sqrt{n+3}+\sqrt{n+2}\right)\left(\sqrt{n+2}+\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n+3}\right)} +$

$\dfrac{2}{\left(\sqrt{n+2}+\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n}+\sqrt{n+2}\right)} =$

$\dfrac{2}{\sqrt{n+2}+\sqrt{n+1}}\left(\dfrac{1}{\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n}+\sqrt{n+2}\right)} - \dfrac{1}{\left(\sqrt{n+3}+\sqrt{n+2}\right)\left(\sqrt{n+1}+\sqrt{n+3}\right)}\right.$

$\dfrac{2\left(\left(\sqrt{n+3}+\sqrt{n+2}\right)\left(\sqrt{n+1}+\sqrt{n+3}\right) - \left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n}+\sqrt{n+2}\right)\right)}{\left(\sqrt{n+3}+\sqrt{n+2}\right)\left(\sqrt{n+2}+\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n+3}\right)\left(\sqrt{n}+\sqrt{n+2}\right)} =$

$\dfrac{2\left(n+\sqrt{(n+1)(n+2)}+\sqrt{(n+1)(n+3)}+\sqrt{(n+2)(n+3)}+3-n-\sqrt{n(n+2)}-\sqrt{n(n+1)}-\sqrt{(n+1)(n+2}\right.}{\left(\sqrt{n+3}+\sqrt{n+2}\right)\left(\sqrt{n+2}+\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n+3}\right)\left(\sqrt{n}+\sqrt{n+2}\right)}$

$\dfrac{2\left(\sqrt{(n+1)(n+3)}+\sqrt{(n+2)(n+3)}-\sqrt{n(n+2)}-\sqrt{n(n+1)}+3\right)}{\left(\sqrt{n+3}+\sqrt{n+2}\right)\left(\sqrt{n+2}+\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n+3}\right)\left(\sqrt{n}+\sqrt{n+2}\right)}$

:

Since $\sqrt{(n+1)(n+3)}+\sqrt{(n+2)(n+3)}-\sqrt{n(n+2)}-\sqrt{n(n+1)}+3 =$

$\sqrt{(n+1)(n+3)}-\sqrt{n(n+1)}+\sqrt{(n+2)(n+3)}-\sqrt{n(n+2)}+3 = \sqrt{n+1}\left(\sqrt{n+3}-\sqrt{n}\right)+\sqrt{n+2}$

$\left(\sqrt{n+1}+\sqrt{n+2}\right)\left(\sqrt{n+3}-\sqrt{n}\right)+3 = \dfrac{3\left(\sqrt{n+1}+\sqrt{n+2}\right)}{\sqrt{n+3}+\sqrt{n}}+3 =$

$\dfrac{3\left(\sqrt{n+3}+\sqrt{n+2}+\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+3}+\sqrt{n}}$ then

$\Delta^3\sqrt{n} = \dfrac{6\left(\sqrt{n+3}+\sqrt{n+2}+\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+3}+\sqrt{n+2}\right)\left(\sqrt{n+2}+\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n+3}\right)\left(\sqrt{n}+\sqrt{n+2}\right)\left(\sqrt{n+3}\right.}$

and, therefore, $\lim\limits_{n\to\infty} n^{5/2}\cdot\Delta^3(\sqrt{n}) =$

$\lim\limits_{n\to\infty}\dfrac{6n^{5/2}\cdot\left(\sqrt{n+3}+\sqrt{n+2}+\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+3}+\sqrt{n+2}\right)\left(\sqrt{n+2}+\sqrt{n+1}\right)\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n+3}\right)\left(\sqrt{n}+\sqrt{n+2}\right)\left(\sqrt{n+3}+\sqrt{n}\right.}$

$\dfrac{6\cdot 4}{64} = \dfrac{3}{8}$