Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

For real $a > 1$ find $\lim\limits_{n\to\infty} \sqrt[n]{\prod_{k=2}^{n}(a - a^{1/k})}$.

Solution 1.

Let $P_n := \prod_{k=2}^{n}(a - a^{1/k}), n \in \mathbb{N}$,where $a > 1$. Noting that by AM-GM inequality

$\left(\frac{1}{a}\right)^{1/n} = \left(\frac{1}{a} \cdot 1 \cdot 1 \cdot \ldots \cdot 1\right)^{1/n} < \dfrac{\frac{1}{a} + (n-1) \cdot 1}{n} = \dfrac{a(n-1)+1}{an}$ we obtain $a^{1/n} > \dfrac{an}{a(n-1)+1}$ and, therefore,

$$a - a^{1/n} < a - \frac{an}{a(n-1)+1} = \frac{(a-1)a(n-1)}{a(n-1)+1} = \frac{2n-2}{2n-1}.$$

Hence, $P_n < \prod_{k=2}^{n} \dfrac{(a-1)a(n-1)}{a(n-1)+1} = (a-1)^{n-1} \prod_{k=1}^{n-1} \dfrac{ak}{ak+1} = (a-1)^{n-1} \prod_{k=1}^{n-1} \dfrac{1}{1 + \frac{1}{ak}}$.

Since by Bernoulli Inequality $\left(1 + \frac{1}{ak}\right)^a > 1 + \frac{1}{ak} \cdot a = \frac{k+1}{k}$ then

$$P_n < (a-1)^{n-1}\left(\prod_{k=1}^{n-1} \frac{1}{\left(1 + \frac{1}{ak}\right)^a}\right)^{1/a} < (a-1)^{n-1}\left(\prod_{k=1}^{n-1} \frac{k}{k+1}\right)^{1/a} = \frac{(a-1)^{n-1}}{n^{1/a}}.$$

Hence $\prod_{k=2}^{n}(a - a^{1/k}) < \dfrac{(a-1)^{n-1}}{n^{1/a}}$.

From the other hand, since by AM-GM Inequality

$(a)^{1/n} = \left(\frac{1}{a} \cdot 1 \cdot 1 \cdot \ldots \cdot 1\right)^{1/n} < \dfrac{a+n-1}{n}$ then $a - a^{1/n} > a - \dfrac{a+n-1}{n} = \dfrac{(a-1)(n-1)}{n}$ and, therefore,

$$P_n = \prod_{k=2}^{n}(a - a^{1/k}) > \prod_{k=2}^{n} \frac{(a-1)(k-1)}{k} = (a-1)^{n-1} \prod_{k=2}^{n} \frac{k-1}{k} = \frac{(a-1)^{n-1}}{n}.$$

Thus, for any $n \in \mathbb{N}$ we have

$$\frac{(a-1)^{n-1}}{n} < P_n < \frac{(a-1)^{n-1}}{n^{1/a}} \Leftrightarrow \frac{\sqrt[n]{(a-1)^{n-1}}}{\sqrt[n]{n}} < \sqrt[n]{P_n} < \frac{\sqrt[n]{(a-1)^{n-1}}}{\sqrt[n]{n^{1/a}}}$$

and since $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{(a-1)^{n-1}}}{\sqrt[n]{n^{1/a}}} = \lim\limits_{n\to\infty} \dfrac{\sqrt[n]{(a-1)^{n-1}}}{\sqrt[n]{n}} = a - 1$

then by Squeeze principle $\lim\limits_{n\to\infty} \sqrt[n]{P_n} = a - 1$.

Solution 2.

Let $a > 1$ and $P_n := \prod_{k=2}^{n}(a - a^{1/k})$. Then $P_n = \dfrac{a^{n-1}}{Q_n}$, where $Q_n := \prod_{k=2}^{n}\left(1 + \dfrac{1}{a^{\frac{k-1}{k}} - 1}\right)$.

Since $\lim\limits_{n\to\infty} \dfrac{Q_{n+1}}{Q_n} = \lim\limits_{n\to\infty}\left(1 + \dfrac{1}{a^{\frac{n}{n+1}} - 1}\right) = 1 + \dfrac{1}{a-1} = \dfrac{a}{a-1}$ then $\lim\limits_{n\to\infty} Q_n^{\frac{1}{n}} = \dfrac{a}{a-1}$ by Multiplicative form of Stolz Theorem (or, by Geometric Mean Limit theorem).

Hence $\lim_{n\to\infty} \sqrt[n]{P_n} = \lim_{n\to\infty} \frac{a^{\frac{n-1}{n}}}{Q_n^{\frac{1}{n}}} = \frac{a}{\frac{a}{a-1}} = a - 1.$