W 36. Calculate $\lim_{n\to\infty}\dfrac{n!\left(1+\frac{1}{n}\right)^{n^2+n}}{n^{n+1/2}}$.

Arkady Alt .

Solution by proposer.

First we will find $\lim_{n\to\infty}\dfrac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n}$.

Since $\ln\left(1+\frac{1}{n}\right)=\frac{1}{n}-\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\iff(n^2+n)\ln\left(1+\frac{1}{n}\right)=$
$n+1-\frac{n+1}{2n}+(n^2+n)o\left(\frac{1}{n^2}\right)n=n+\frac{1}{2}-\frac{1}{2n}+n^2o\left(\frac{1}{n^2}\right)$

then $\lim_{n\to\infty}\dfrac{\left(1+\frac{1}{n}\right)^{n^2+n}}{e^n}=e^{\lim_{n\to\infty}\left((n^2+n)\ln\left(1+\frac{1}{n}\right)-n\right)}=e^{\lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{2n}+n^2o\left(\frac{1}{n^2}\right)\right)}=e^{\frac{1}{2}}$.

Thus, $\lim_{n\to\infty}\dfrac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n}=e^{\frac{1}{2}}$.

Using obtained asymptotic equivalence $\left(1+\frac{1}{n}\right)^{n^2+n}\sim e^{n+1/2}$ and Stirling asymptotic equivalence $n!\sim\left(\frac{n}{e}\right)^n\sqrt{2\pi n}$ we obtain $\lim_{n\to\infty}\dfrac{n!\left(1+\frac{1}{n}\right)^{n^2+n}}{n^{n+1/2}}=$

$$\lim_{n\to\infty}\frac{n!}{\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}\cdot\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n^2+n}\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}{n^{n+1/2}}=$$
$$\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n^2+n}\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}{n^{n+1/2}}=\sqrt{2\pi}\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n^2+n}}{e^n}=$$
$$\sqrt{2\pi}\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n^2+n}e^{1/2}}{e^{n+1/2}}=\sqrt{2\pi e}.$$

Remark. Another way to calculate $\lim_{n\to\infty}\dfrac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n}$.

Since $\lim_{n\to\infty}\dfrac{\left(1+\frac{1}{n}\right)}{e}=1$ then $\lim_{n\to\infty}\dfrac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n}=\lim_{n\to\infty}\left(\dfrac{\left(\left(1+\frac{1}{n}\right)\right)^n}{e}\right)^n=\lim_{n\to\infty}(1+\alpha_n)^n$,

where $\alpha_n:=\dfrac{\left(\left(1+\frac{1}{n}\right)\right)^n}{e}-1$ and $\lim_{n\to\infty}\alpha_n=0$.

We have $\lim_{n\to\infty}(1+\alpha_n)^n=\lim_{n\to\infty}\left((1+\alpha_n)^{1/\alpha_n}\right)^{n\alpha_n}$ and the problem reduced to calculation of

$\lim_{n\to\infty}n\alpha_n=\lim_{n\to\infty}n\left(\dfrac{\left(\left(1+\frac{1}{n}\right)\right)^n}{e}-1\right)=\lim_{n\to\infty}n(e^{\beta_n}-1)$, where

$\beta_n:=\ln\left(\dfrac{\left(\left(1+\frac{1}{n}\right)\right)^n}{e}\right)=$

$n\ln\left(1+\frac{1}{n}\right)-1$ and $\lim_{n\to\infty}\beta_n=0$.

Note that $\lim_{n\to\infty} n(e^{\beta_n}-1) = \lim_{n\to\infty}\left(n\beta_n \cdot \frac{e^{\beta_n}-1}{\beta_n}\right)$ and $\lim_{n\to\infty}\frac{e^{\beta_n}-1}{\beta_n} = 1$. Thus, suffice to find
$\lim_{n\to\infty} n\beta_n = \lim_{n\to\infty} n\left(n\ln\left(1+\frac{1}{n}\right)-1\right) = \lim_{n\to\infty}\left(n^2\ln\left(1+\frac{1}{n}\right)-n\right)$.
Since $\frac{1}{n}-\frac{1}{2n^2} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}$ then
$n^2\left(\frac{1}{n}-\frac{1}{2n^2}\right)-n < n^2\ln\left(1+\frac{1}{n}\right)-n < n^2\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\right)-n \iff$
$-\frac{1}{2} < n^2\ln\left(1+\frac{1}{n}\right)-n < -\frac{1}{2}+\frac{1}{3n}$ and, therefore, by squeeze principle
$\lim_{n\to\infty} n\alpha_n = \lim_{n\to\infty} n(e^{\beta_n}-1) = \lim_{n\to\infty} n\beta_n \cdot \lim_{n\to\infty}\frac{e^{\beta_n}-1}{\beta_n} = \lim_{n\to\infty} n\beta_n = -\frac{1}{2}$.
Since $\lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n}\right)^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$ then $\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n^2+n}}{e^n} =$
$\lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n}\right)^{n+1}}{e}\right)^n = \lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n}\right)^n}{e}\right)^n \cdot \left(\frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n}\right)^n = e^{-1/2}\cdot e = e^{1/2}$.
Thus, $\left(1+\frac{1}{n}\right)^{n^2+n} \sim e^{n+1/2}$.