Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Let ABC be a non-obtuse triangle with usual notations. Prove that

$$\frac{2r}{R} + \frac{r^2}{R^2} \le \sum \frac{a^2 \cos A}{bc} \le 1 + \frac{r}{R}.$$

Solution.

Proof of $\sum \frac{a^2 \cos A}{bc} \le 1 + \frac{r}{R}$.

$\sum a^3 \cos A = \sum a^2 \cos A \cdot a = \sum a^2 \cos A(b \cos C + c \cos B) =$
$\sum a^2 b \cos A \cos C + \sum a^2 c \cos A \cos B = \sum a^2 b \cos A \cos C + \sum b^2 a \cos B \cos C =$
$\sum ab \cos C(a \cos A + b \cos B) \le \sum ab \cos C(a \cos B + b \cos A) = \sum abc \cos C =$
$abc \sum \cos C = abc\left(1 + \frac{r}{R}\right) \le \frac{3}{2} abc.$

Proof of $\frac{2r}{R} + \frac{r^2}{R^2} \le \sum \frac{a^2 \cos A}{bc}$.

$\sum a^3 \cos A = \sum a^2 \cos A \cdot a = \sum a^2 \cos A(b \cos C + c \cos B) =$
$\sum a^2 b \cos A \cos C + \sum a^2 c \cos A \cos B = \sum b^2 c \cos B \cos A + \sum a^2 c \cos A \cos B =$
$\sum (a^2 + b^2) c \cos A \cos B \ge 2abc \sum \cos A \cos B = 2abc \cdot \frac{s^2 + r^2 - 4R^2}{4R^2} = abc \cdot \frac{s^2 + r^2 - 4R^2}{2R^2}.$

Since in any non-obtuse triangle holds inequality $2R + r \le s$ (with equality in right-angled triangle) then $\frac{s^2 + r^2 - 4R^2}{2R^2} \ge \frac{(2R + r)^2 + r^2 - 4R^2}{2R^2} = \frac{(2R + r)r}{R^2}$ and, therefore,

we obtain $\sum \cos A \cos B \ge \frac{(2R + r)r}{R^2}$.

Thus, $abc \cdot \frac{(2R + r)r}{R^2} \le \sum a^3 \cos A \le abc\left(1 + \frac{r}{R}\right) \Leftrightarrow$

$$\frac{2r}{R} + \frac{r^2}{R^2} \le \sum \frac{a^2 \cos A}{bc} \le 1 + \frac{r}{R}.$$