Problem with a solution proposed by Arkady Alt , San Jose, California, USA
Infinite sum for sequence defined by polynomial recurrence.

Let $p \geq 2$ be positive integer and $a > 1$ be real number. Find

$$\sum_{n=0}^{\infty} \frac{a_n^{p-2} + 2a_n^{p-3} + 3a_n^{p-4} + \ldots + p - 1}{1 + a_n + a_n^2 + \ldots + a_n^{p-1}},$$

where $a_{n+1} = \dfrac{1}{p}(a_n^p + p - 1), n \in \mathbb{N} \cup \{0\}$ and $a_0 = a > 1$.

Solution.

First note that $a_{n+1} = \dfrac{1}{p}(a_n^p + p - 1) \iff a_{n+1} - 1 = \dfrac{1}{p}(a_n^p - 1)$.

Also note that

$$\frac{1}{x-1} - \frac{x^2 + 2x + 3}{1 + x + x^2 + x^3} = \frac{4}{x^4 - 1}$$

$$\frac{1}{x-1} - \frac{x^3 + 2x^2 + 3x + 4}{1 + x + x^2 + x^3 + x^4} = \frac{5}{x^5 - 1}$$

and more general

(**1**) $$\frac{1}{x-1} - \frac{x^{p-2} + 2x^{p-3} + 3x^{p-4} + \ldots + p - 1}{1 + x + x^2 + x^3 + \ldots + x^{p-1}} = \frac{p}{x^p - 1}, p \geq 2.$$

Indeed, $\sum_{k=0}^{p-1} x^k - (x-1)\sum_{k=0}^{p-2}(k+1)x^{p-2-k} = \sum_{k=0}^{p-1} x^k - (x-1)\sum_{k=0}^{p-2}(p-1-k)x^k =$

$\sum_{k=0}^{p-1} x^k - (x-1)\sum_{k=1}^{p-1}(p-k)x^{k-1}$ and, since $(x-1)\sum_{k=1}^{p-1}(p-k)x^{k-1} = \sum_{k=1}^{p-1}(p-k)x^k - \sum_{k=1}^{p-1}(p-k)x^{k-1} =$

$\sum_{k=1}^{p-1}(p-k)x^k - \sum_{k=0}^{p-2}(p-k-1)x^k = \sum_{k=1}^{p-2}(p-k)x^k - \sum_{k=1}^{p-2}(p-k-1)x^k + (p-p+1)x^{p-1} - (p-1) =$

$\sum_{k=1}^{p-2} x^k + x^{p-1} - (p-1) = \sum_{k=0}^{p-1} x^k - p$ then $\sum_{k=0}^{p-1} x^k - (x-1)\sum_{k=1}^{p-1}(p-k)x^{k-1} = p.$

Let $S_n := \sum_{k=0}^{n} \dfrac{a_k^{p-2} + 2a_k^{p-3} + 3a_k^{p-4} + \ldots + p - 1}{1 + a_k + a_k^2 + \ldots + a_k^{p-1}}, n \in \mathbb{N}.$

Applying identity (**1**) we will prove using Math. Induction that

$$\frac{1}{a-1} - S_n = \frac{1}{a_{n+1} - 1}, n \in \mathbb{N} \cup \{0\}.$$

1. Base of Math. Induction.

Let $n = 0$. Since $a_0 = a$ then

$$\frac{1}{a-1} - S_0 = \frac{1}{a-1} - \frac{a^{p-2} + 2a^{p-3} + 3a^{p-4} + \ldots + p - 1}{1 + a + a^2 + a^3 + \ldots + a^{p-1}} = \frac{p}{a^p - 1} = \frac{1}{\dfrac{a_0^p - 1}{p}} = \frac{1}{a_1 - 1}.$$

2. Step of Math Induction.

Assuming $\dfrac{1}{a-1} - S_n = \dfrac{1}{a_{n+1} - 1}$ and using (1) we obtain $\dfrac{1}{a-1} - S_{n+1} =$

$$\frac{1}{a-1} - S_n - \frac{a_{n+1}^{p-2} + 2a_{n+1}^{p-3} + 3a_{n+1}^{p-4} + \ldots + p - 1}{1 + a_{n+1} + a_{n+1}^2 + \ldots + a_{n+1}^{p-1}} = \frac{1}{a_{n+1} - 1} - \frac{a_{n+1}^{p-2} + 2a_{n+1}^{p-3} + 3a_{n+1}^{p-4} + \ldots + p - 1}{1 + a_{n+1} + a_{n+1}^2 + \ldots + a_{n+1}^{p-1}} =$$

$$\frac{p}{a_{n+1}^p - 1} = \frac{1}{\dfrac{a_{n+1}^p - 1}{p}} = \frac{1}{a_{n+2} - 1}.$$

Let $t := a - 1$. Then $a = t + 1, t > 0$ and using Math. Induction we will prove inequality

(2) $a_n \geq 1 + t + \frac{(p-1)n}{2}t^2, n \in \mathbb{N}.$

Applying inequality*

(3) $(1+x)^p \geq 1 + px + \frac{p(p-1)}{2}x^2, x > 0$ and $p \in \mathbb{N}$

to $x = t + \frac{(p-1)n}{2}t^2$ we obtain:

1. Base of Math. Induction.

$$a_1 = \frac{1}{p}(a^p + p - 1) = \frac{1}{p}((1+t)^p + p - 1) \geq \frac{1}{p}\left(1 + pt + \frac{p(p-1)}{2}t^2 + p - 1\right) = 1 + t + \frac{(p-1)n}{2}t^2.$$

2. For any $n \in \mathbb{N}$, noting that

$$\left(1 + t + \frac{(p-1)n}{2}t^2\right)^p \geq 1 + p\left(t + \frac{(p-1)n}{2}t^2\right) + \frac{(p-1)n}{2}\left(t + \frac{(p-1)n}{2}t^2\right)^2 =$$
$$1 + pt + \frac{p(p-1)n}{2}t^2 + \frac{p(p-1)}{2}\left(t^2 + (p-1)nt^3 + \frac{(p-1)^2n^2}{4}t^4\right) >$$

$$1 + pt + \frac{p(p-1)n}{2}t^2 + \frac{p(p-1)}{2}t^2 = 1 + pt + \frac{p(p-1)(n+1)}{2}t^2$$

and assuming $a_n \geq 1 + t + \frac{(p-1)n}{2}t^2$, we obtain

$$a_{n+1} = \frac{1}{p}(a_n^p + p - 1) \geq \frac{1}{p}\left(\left(1 + t + \frac{(p-1)n}{2}t^2\right)^p + p - 1\right) >$$
$$\frac{1}{p}\left(1 + pt + \frac{p(n+1)(p-1)}{2}t^2 + p - 1\right) = 1 + t + \frac{(p-1)(n+1)}{2}t^2. \blacksquare$$

Hence, $0 < \frac{1}{a_{n+1} - 1} < \frac{1}{t + \frac{(p-1)(n+1)}{2}t^2}$ and since $\lim_{n\to\infty} \frac{1}{t + \frac{(p-1)(n+1)}{2}t^2} = 0$

then $\lim_{n\to\infty} \frac{1}{a_{n+1} - 1} = 0$ and, therefore, $\lim_{n\to\infty} S_n = \frac{1}{a-1} - \lim_{n\to\infty} \frac{1}{a_{n+1} - 1} = \frac{1}{a-1}.$