Asymptotic behavior 6.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

Calculate $\lim\limits_{n\to\infty} \dfrac{n!\left(1+\frac{1}{n}\right)^{n^2+n}}{n^{n+1/2}}$.

Solution.

First we will find* $\lim\limits_{n\to\infty} \dfrac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n}$.

Since $\ln\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) \iff (n^2+n)\ln\left(1+\frac{1}{n}\right) =$

$n+1-\frac{n+1}{2n} + (n^2+n)o\left(\frac{1}{n^2}\right)n = n+\frac{1}{2}-\frac{1}{2n}+n^2 o\left(\frac{1}{n^2}\right)$

then $\lim\limits_{n\to\infty} \dfrac{\left(1+\frac{1}{n}\right)^{n^2+n}}{e^n} = e^{\lim\limits_{n\to\infty}\left((n^2+n)\ln\left(1+\frac{1}{n}\right)-n\right)} = e^{\lim\limits_{n\to\infty}\left(\frac{1}{2}-\frac{1}{2n}+n^2 o\left(\frac{1}{n^2}\right)\right)} = e^{\frac{1}{2}}$.

Thus, $\lim\limits_{n\to\infty} \dfrac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n} = e^{\frac{1}{2}}$.)

Using Stirling asymptotic equivalence $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ and $\left(1+\frac{1}{n}\right)^{n^2+n} \sim e^{n+1/2}$

we obtain $\lim\limits_{n\to\infty} \dfrac{n!\left(1+\frac{1}{n}\right)^{n^2+n}}{n^{n+1/2}} = \lim\limits_{n\to\infty} \dfrac{n!}{\left(\frac{n}{e}\right)^n\sqrt{2\pi n}} \cdot \lim\limits_{n\to\infty} \dfrac{\left(1+\frac{1}{n}\right)^{n^2+n}\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}{n^{n+1/2}} =$

$\lim\limits_{n\to\infty} \dfrac{\left(1+\frac{1}{n}\right)^{n^2+n}\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}{n^{n+1/2}} = \sqrt{2\pi}\lim\limits_{n\to\infty} \dfrac{\left(1+\frac{1}{n}\right)^{n^2+n}}{e^n} =$

$\sqrt{2\pi}\lim\limits_{n\to\infty} \dfrac{\left(1+\frac{1}{n}\right)^{n^2+n} e^{1/2}}{e^{n+1/2}} = \sqrt{2\pi e}$.

* Another way to find $\lim\limits_{n\to\infty} \dfrac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n}$

Since $\lim\limits_{n\to\infty} \dfrac{\left(1+\frac{1}{n}\right)}{e} = 1$ then $\lim\limits_{n\to\infty} \dfrac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n} = \lim\limits_{n\to\infty}\left(\dfrac{\left(\left(1+\frac{1}{n}\right)\right)^n}{e}\right)^n = \lim\limits_{n\to\infty}(1+\alpha_n)^n$,

where $\alpha_n := \dfrac{\left(\left(1+\frac{1}{n}\right)\right)^n}{e} - 1$ and $\lim\limits_{n\to\infty}\alpha_n = 0$.

We have $\lim\limits_{n\to\infty}(1+\alpha_n)^n = \lim\limits_{n\to\infty}\left((1+\alpha_n)^{1/\alpha_n}\right)^{n\alpha_n}$ and the problem reduced to calculation of

$\lim\limits_{n\to\infty} n\alpha_n = \lim\limits_{n\to\infty} n\left(\dfrac{\left(\left(1+\frac{1}{n}\right)\right)^n}{e} - 1\right) = \lim\limits_{n\to\infty} n(e^{\beta_n}-1)$, where

$\beta_n := \ln\left(\dfrac{\left(\left(1+\frac{1}{n}\right)\right)^n}{e}\right) =$

$n\ln\left(1+\frac{1}{n}\right) - 1$ and $\lim\limits_{n\to\infty}\beta_n = 0$.

Note that $\lim\limits_{n\to\infty} n(e^{\beta_n}-1) = \lim\limits_{n\to\infty}\left(n\beta_n \cdot \dfrac{e^{\beta_n}-1}{\beta_n}\right)$ and $\lim\limits_{n\to\infty}\dfrac{e^{\beta_n}-1}{\beta_n} = 1$. Thus, suffice to find

$\lim\limits_{n\to\infty} n\beta_n = \lim\limits_{n\to\infty} n\left(n\ln\left(1+\frac{1}{n}\right)-1\right) = \lim\limits_{n\to\infty}\left(n^2\ln\left(1+\frac{1}{n}\right)-n\right).$

Since $\frac{1}{n}-\frac{1}{2n^2} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}$ then

$n^2\left(\frac{1}{n}-\frac{1}{2n^2}\right)-n < n^2\ln\left(1+\frac{1}{n}\right)-n < n^2\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\right)-n \iff$

$-\frac{1}{2} < n^2\ln\left(1+\frac{1}{n}\right)-n < -\frac{1}{2}+\frac{1}{3n}$ and, therefore, by squeeze principle

$\lim\limits_{n\to\infty} n\alpha_n = \lim\limits_{n\to\infty} n(e^{\beta_n}-1) = \lim\limits_{n\to\infty} n\beta_n \cdot \lim\limits_{n\to\infty}\frac{e^{\beta_n}-1}{\beta_n} = \lim\limits_{n\to\infty} n\beta_n = -\frac{1}{2}.$

Since $\lim\limits_{n\to\infty}\left(\frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}}\right)^{n} = \lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^{n} = e$ then $\lim\limits_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n^2+n}}{e^n} =$

$\lim\limits_{n\to\infty}\left(\frac{\left(1+\frac{1}{n}\right)^{n+1}}{e}\right)^{n} = \lim\limits_{n\to\infty}\left(\frac{\left(1+\frac{1}{n}\right)^{n}}{e}\right)^{n}\cdot\left(\frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}}\right)^{n} = e^{-1/2}\cdot e = e^{1/2}.$

Thus, $\left(1+\frac{1}{n}\right)^{n^2+n} \sim e^{n+1/2}.$