Infinum of product of two binomials in power exponential domain.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

Find $\inf\limits_{(x,y)\in D}(x-1)(y-1)$ where $D := \{(x,y) \mid x,y \in \mathbb{R}_+,\ x \neq y \text{ and } x^y = y^x\}$.

Solution.

First note that $x \neq 1$ and $y \neq 1$ because otherwise we get $x = y = 1$.

Let $D_< := \{(x,y) \mid x,y \in \mathbb{R}_+,\ x < y \text{ and } x^y = y^x\}$. Due to symmetry we have $\inf\limits_{(x,y)\in D}(x-1)(y-1) = \inf\limits_{(x,y)\in D_<}(x-1)(y-1)$.

Let $f(x) := x^{\frac{1}{x}}, x > 0$. Since $f'(x) = \dfrac{f(x)}{x^2}(1-\ln x)$ then $f(x)$ strictly increasing on $(0,e]$ and strictly decreasing on $[0,\infty)$ with $\max\limits_{x>0} f(x) = f(e) = e^{\frac{1}{e}}$.

Therefore, noting that $x^y = y^x \iff x^{\frac{1}{x}} = y^{\frac{1}{y}} \iff f(x) = f(y)$, we can conclude that $(x,y) \in D_< \Rightarrow x < e < y$ (otherwise if x,y both belong to $(0,e)$ or (e,∞) then, due to monotonicity of $f(x)$, equality $f(x) = f(y)$ yields $x = y$, that is the contradiction.

And in case $x = e$, since $y \neq e$ we again obtain contradiction $f(e) = f(y) \neq f(e)$).

Also note that if $(x,y) \in D_<$ then $x > 1$. Indeed, since $x^y = y^x \iff x = y\log_y x$ and $y > e$ then supposition $x < 1$ implies $0 < \frac{x}{y} = \log_y x < 0$, i.e. contradiction.

Hence $(x,y) \in D_< \Rightarrow x \in (1,e),\ y \in (e,\infty)$.

Let $t := \log_x y - 1$. Then $\log_x y = t+1 \iff y = x^{t+1}$ and $y = x\log_x y \iff y = x(t+1)$. By substitution $y = x(t+1)$ in $y^x = x^y \iff \frac{y}{x} = x^{\frac{y}{x}-1}$ we obtain $t+1 = x^t \iff x = (t+1)^{\frac{1}{t}} \Rightarrow y = (t+1)^{1+\frac{1}{t}}$, where $t > 0$ (since $1 < x < y$).

Thus, $D_< = \left\{\left((t+1)^{\frac{1}{t}}, (t+1)^{1+\frac{1}{t}}\right) \mid t \in (0,\infty)\right\}$ is set of all non-trivial solution of equation $x^y = y^x$, satisfied $x < y$.

Let $H(t) := (x-1)(y-1) = (x-1)(x(t+1)-1) = x(x-1)(t+1) - x + 1$.

We have $H'(t) = (2x-1)x'(t+1) + x^2 - x - x' = x'((2x-1)(t+1)-1) + x^2 - x =$
$x'(2x(t+1)-(t+2)) + x^2 - x = x(2x(t+1)-(t+2))(\ln x)' + x^2 - x =$
$x\left((2x(t+1)-(t+2))\left(\dfrac{\ln(1+t)}{t}\right)' + x - 1\right) =$
$x\left((2x(t+1)-(t+2))\left(\dfrac{1}{t(1+t)} - \dfrac{\ln(1+t)}{t^2}\right) + x - 1\right).$

Since by Bernoulli Inequality $(1+t)^{1+\frac{1}{t}} > 1 + (1+\frac{1}{t})t = 2+t$ then $2(1+t)^{1+\frac{1}{t}} - (t+2) > 4 + 2t - t - 2 > 0$ and also, since $\ln(1+t) < t$, we have

$$\left(\frac{1}{t(1+t)} - \frac{\ln(1+t)}{t^2}\right) + (t+1)^{\frac{1}{t}} - 1 > \left(\frac{1}{t(1+t)} - \frac{t}{t^2}\right) + (t+1)^{\frac{1}{t}} - 1 =$$
$$\left(\frac{1}{t(1+t)} - \frac{1}{t}\right) + (t+1)^{\frac{1}{t}} - 1 = \frac{1}{1+t} + (t+1)^{\frac{1}{t}} - 1 = (t+1)^{\frac{1}{t}} - \frac{t}{1+t} =$$
$$\frac{(t+1)^{1+\frac{1}{t}} - t}{1+t} > \frac{1 + t(1+\frac{1}{t}) - t}{1+t} = \frac{2}{1+t} > 0.$$

So, $H(t)$ increasing on $(0,\infty)$ and, therefore, $(x-1)(y-1) = H(t) > \lim\limits_{t\to 0} H(t) = (e-1)^2$.

Hence, $\inf_{(x,y)\in D} (x-1)(y-1) = (e-1)^2$.