Asymptotic behavior-General case.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

Let $P_m(x) := \sum_{k=0}^{m} \frac{(-1)^k x^{m-k}}{k+1}, m \in \mathbb{N}$. For any $m \in \mathbb{N}$ calculate $\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^{n^m}}{e^{P_{m-1}(n)}}$.

Solution.

Noting that $\ln\left(1+\frac{1}{n}\right) = \sum_{k=1}^{m} \frac{(-1)^{k-1}}{kn^k} + o\left(\frac{1}{n^m}\right)$ we obtain that $n^m \ln\left(1+\frac{1}{n}\right) =$
$n^m \sum_{k=1}^{m} \frac{(-1)^{k-1}}{kn^k} + n^m o\left(\frac{1}{n^m}\right) = \sum_{k=1}^{m} \frac{(-1)^{k-1} n^{m-k}}{k} + n^m o\left(\frac{1}{n^m}\right) =$
$\sum_{k=0}^{m-1} \frac{(-1)^k n^{m-1-k}}{k+1} + n^m o\left(\frac{1}{n^m}\right) = P_{m-1}(n) + n^m o\left(\frac{1}{n^m}\right)$

Since $n^m \ln\left(1+\frac{1}{n}\right) - P_{m-1}(n) = n^m o\left(\frac{1}{n^m}\right)$ then $\lim_{n\to\infty}\left(n^m \ln\left(1+\frac{1}{n}\right) - P_{m-1}(n)\right) = 0$

and, therefore, $\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^{n^m}}{e^{P_{m-1}(n)}} = e^{\lim_{n\to\infty}\left(n^m \ln\left(1+\frac{1}{n}\right) - P_{m-1}(n)\right)} = 1.$

Thus, $\left(1+\frac{1}{n}\right)^{n^m} \sim e^{P_{m-1}(n)}$.