$$T_{2n+1} := \frac{u^{2n+1}-v^{2n+1}}{u-v}+\frac{v^{2n+1}-w^{2n+1}}{v-w}+\frac{w^{2n+1}-u^{2n+1}}{w-u}$$

We have successively,

$$T_{2n+1}=\left(u^{2n}+u^{2n-1}v+uv^{2n-1}+v^{2n}\right)+\left(v^{2n}+v^{2n-1}w+vw^{2n-1}+w^{2n}\right)+$$

$$+\left(w^{2n}+w^{2n-1}u+wu^{2n-1}+u^{2n}\right)=$$

$$=2S^{(2n)}+\sum_{k=1}^{2n}\left(u^{2n-k}v^{k}+v^{2n-k}w^{k}+w^{2n-k}u^{k}\right)=$$

$$=2S^{(2n)}+\sum_{k=1}^{n}(u^{2n-k}v^{k}+u^{k}v^{2n-k}+v^{2n-k}w^{k}+v^{k}w^{2n-k}+$$

$$+w^{2n-k}u^{k}+w^{k}u^{2n-k})=2S^{(2n)}+\sum_{k=1}^{n}[(u^{2n-k}+v^{2n-k}+w^{2n-k})\cdot$$

$$\cdot\left(u^{k}+v^{k}+w^{k}\right)-(u^{2n}+v^{2n}+w^{2n})]=$$

$$2S^{(2n)}+\sum_{k=1}^{n}\left(S^{(2n-k)}S^{(k)}-S^{(2n)}\right)=(2-n)\,S^{(2n)}+\sum_{k=1}^{n}S^{(2n-k)}S^{(k)}$$

How the sums $S^{(k)}$ are integers, then the sum T_{2n+1} is also an integer.

W28. Solution by the proposer. Since

$$F_{n+1}\cdot F_{n-1}-F_n^2=(-1)^n \text{ (Cassini identity)}$$

then for

$$(x_0,y_0)=((-1)^n F_{n-1},(-1)^n F_n)$$

we have

$$F_{n+1}x_0-F_ny_0=F_{n+1}\cdot(-1)^n F_{n-1}-F_n\cdot(-1)^n F_n=$$

$$=(-1)^n\left(F_{n+1}\cdot F_{n-1}-F_n^2\right)=1$$

and, therefore

$$F_{n+1}x - F_n y = 1 \Leftrightarrow F_{n+1}x - F_n y = F_{n+1}\cdot(-1)^n F_{n-1} - F_n \cdot (-1)^n F_n \Leftrightarrow$$

$$\Leftrightarrow F_{n+1}\left(x - (-1)^n F_{n-1}\right) = F_n\left(y - (-1)^n F_n\right) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x - (-1)^n F_{n-1} = tF_n \\ y - (-1)^n F_{n-1} = tF_{n+1} \end{cases} \Leftrightarrow \begin{cases} x = tF_n + (-1)^n F_{n-1} \\ y = tF_{n+1} + (-1)^n F_n \end{cases}$$

$t \in Z$. Thus

$$D_n = \{(x,y)\,|x = tF_n + (-1)^n F_{n-1}, y = tF_{n+1} + (-1)^n F_n, t \in Z\}$$

a).

$$\min_{(x,y)\in D_n} |x+y| = \min_{t\in Z} |t(F_n + F_{n+1}) + (-1)^n (F_{n-1} + F_n)| = \min_{t\in Z} \varphi(t)$$

where $\varphi(t) := |tF_{n+2} + (-1)^n F_{n+1}|$.
Since

$$tF_{n+2} + (-1)^n F_{n+1} = 0 \Leftrightarrow t = t_* := \frac{(-1)F_{n+1}}{F_{n+2}}$$

and

$$|t_*| = \frac{F_{n+1}}{F_{n+2}} < 1$$

then integer t which minimize $\varphi(t)$ must be among to t_* integers, that is.
Thus

$$\min_{t\in Z} \varphi(t) = \min_{t\in\{-1,0,1\}} \varphi(t) = \min\{\varphi(0), \varphi(1), \varphi(-1)\} =$$

$$= \min\{F_{n+1}, |F_{n+2} + (-1)^n F_{n+1}|, |-F_{n+2} + (-1)^n F_{n+1}|\}$$

If n is odd then

$$\min\{F_{n+1}, |F_{n+2} + (-1)^n F_{n+1}|, |-F_{n+2} + (-1)^n F_{n+1}|\} =$$

$$= \min\{F_{n+1}, F_{n+2} - F_{n+1}, F_{n+2} + F_{n+1}\} =$$

$$= \min\left\{F_{n+1}, F_n, F_{n+3}\right\} = \varphi(1)$$

If n is even then

$$\min\left\{F_{n+1}, F_{n+2} + F_{n+1}, \left|-F_{n+2} + F_{n+1}\right|\right\} =$$

$$= \min\left\{F_{n+1}, F_{n+3}, F_n\right\} = \varphi(-1)$$

So,

$$\min_{(x,y)\in D_n} |x+y| = F_n$$

b).

$$|x| + |y| = \left|tF_n + (-1)^n F_{n-1}\right| + \left|tF_{n+1} + (-1)^n F_n\right|$$

By replacing t with $(-1)^{n-1} t$ we obtain

$$|x| + |y| = \left|tF_n - F_{n-1}\right| + \left|tF_{n+1} - F_n\right| =$$

$$\max\left\{\left|tF_n - F_{n-1} + tF_{n+1} - F_n\right|, \left|tF_n - F_{n-1} - tF_{n+1} + F_n\right|\right\} =$$

$$= \max\left\{\left|t\left(F_n + F_{n+1}\right) - \left(F_{n-1} + F_n\right)\right|, \left|t\left(F_n - F_{n+1}\right) + F_n - F_{n-1}\right|\right\} =$$

$$= \max\left\{\left|tF_{n+2} - F_{n+1}\right|, \left|-F_{n-1}t + F_{n-2}\right|\right\} =$$

$$= \max\left\{\left|tF_{n+2} - F_{n+1}\right|, \left|F_{n-1}t - F_{n-2}\right|\right\}$$

Lemma. Let $p, q > 0$, $a, b \in R$. Then

$$\min_{x\in R}\left(\max\left\{|px - a|, |qx - b|\right\}\right) = \frac{|aq - bp|}{p+q}$$

and attained if $x = \frac{a+b}{p+q}$.

Proof. First note $\min\limits_{x\in R}\left(\max\left\{|px - a|, |qx - b|\right\}\right) = \min t$ where t provide solvability of inequality

$$t \geq \max\left\{|px-a|, |qx-b|\right\} \Leftrightarrow \left\{\begin{array}{l} t \geq |px-a| \\ t \geq |qx-b| \end{array}\right. \Leftrightarrow$$

$$\Leftrightarrow \left\{\begin{array}{l} -t \leq px-a \leq t \\ -t \leq qx-b \leq t \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \frac{a-t}{p} \leq x \leq \frac{t+a}{p} \\ \frac{b-t}{q} \leq x \leq \frac{t+b}{q} \end{array}\right. \Leftrightarrow$$

$$\Leftrightarrow \max\left\{\frac{a-t}{p}, \frac{b-t}{q}\right\} \leq x \leq \min\left\{\frac{t+a}{p}, \frac{t+b}{q}\right\} \tag{1}$$

Condition of solvability is

$$\left\{\begin{array}{l} \frac{a-t}{p} \leq \frac{t+b}{q} \\ \frac{b-t}{q} \leq \frac{t+a}{p} \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} q(a-t) \leq p(t+b) \\ p(b-t) \leq q(t+a) \end{array}\right. \Leftrightarrow$$

$$\Leftrightarrow \left\{\begin{array}{l} \frac{aq-pb}{p+q} \leq t \\ \frac{pb-aq}{p+q} \leq t \end{array}\right. \Leftrightarrow \frac{|aq-bp|}{p+q} \leq t$$

Thus

$$\min_{x \in R}\left(\max\left\{|px-a|, |qx-b|\right\}\right) = \frac{|aq-bp|}{p+q}$$

Assuming WLOG that $aq \geq bp$, by substitution $t = \frac{aq-pb}{p+q}$ in inequality (1) we obtain $x = \frac{a+b}{p+q}$. Indeed,

$$\frac{t+a}{p} = \frac{\frac{aq-pb}{p+q}+a}{p} = \frac{2aq+ap-bp}{p(p+q)}, \frac{t+b}{q} = \frac{\frac{aq-pb}{p+q}+b}{q} = \frac{a+b}{p+q}$$

and

$$\frac{2aq+ap-bp}{p(p+q)} - \frac{a+b}{p+q} = \frac{2}{p} \cdot \frac{aq-pb}{p+q} \geq 0$$

Hence $x \leq \frac{a+b}{p+q}$. Similarly we can obtain $\frac{a+b}{p+q} \leq x$. So, $x = \frac{a+b}{p+q}$.

Corollary. Let $p > q$ then

$$\min_{x \in R}\left(|px-a| + |qx-b|\right) = \frac{|aq-bp|}{p+q}$$

Proof. Since

$$|px-a|+|qx-b| = \max\{|(p+q)x-(a+b)|, |(p-q)x-(a-b)|\}$$

then by Lemma

$$|px-a|+|qx-b| \geq \frac{|(a+b)(p-q)-(a-b)(p+q)|}{(p+q)+(p-q)} = \frac{|aq-bp|}{p}$$

and equality occurs iff

$$x = \frac{(a+b)+(a-b)}{(p+q)+(p-q)} = \frac{2a}{2p} = \frac{a}{p}$$

So,

$$\min_{x\in R}(|px-a|+|qx-b|) = \frac{|aq-bp|}{p}$$

and attained if $x = \frac{a}{p}$.

Application. Since

$$|x|+|y| = |tF_n - F_{n+1}| + |tF_{n+1} - F_n|$$

and $F_{n+1} > F_n$ then by Corollary for real t minimum of $|tF_n - F_{n+1}| + |tF_{n+1} - F_n|$ attained if $t_* = \frac{F_n}{F_{n+1}} < 1$ and closest to t_* integer values of t are $t = 1$ and $t = 0$. Therefore,

$$\min_{t\in Z}|tF_n - F_{n+1}| + |tF_{n+1} - F_n| =$$

$$= \min\{|0\cdot F_n - F_{n+1}| + |0\cdot F_{n+1} - F_n|, |1\cdot F_n - F_{n+1}| + |1\cdot F_{n+1} - F_n|\} =$$

$$= \min\{F_{n-1}+F_n, F_n - F_{n-1} + F_{n+1} - F_n\} = \min\{F_{n+1}, F_n\} = F_n$$

c).

$$x^2+y^2 = (tF_n - F_{n-1})^2 + (tF_{n+1} - F_n)^2 =$$

$$= \left(F_n^2 + F_{n+1}^2\right)t^2 - 2F_n(F_{n-1}+F_{n+1})t + \left(F_n^2 + F_{n-1}^2\right)$$

Since

$$F_{n-1}^2 + F_n^2 = F_{2n-1} \text{ and } F_n\left(F_{n+1} + F_{n-1}\right) = F_{2n}$$

then

$$x^2 + y^2 = F_{2n+1}t^2 - 2F_{2n}t + F_{2n-1}$$

Quadratic trinomial $F_{2n+1}t^2 - 2F_{2n}t + F_{2n-1}$ attain minimum in R when $t = \frac{F_{2n}}{F_{2n+1}} < 1$ and, therefore, minimum in Z can be attained in one of two integer points closest to $\frac{F_{2n}}{F_{2n+1}}$, that is when $t = 0$ or $t = 1$.
Thus

$$\min\left(x^2 + y^2\right) =$$

$$= \min\left\{F_{2n+1} \cdot 0^2 - 2F_{2n} \cdot 0 + F_{2n-1}, F_{2n+1} \cdot 1^2 - 2F_{2n} \cdot 1 + F_{2n-1}\right\} =$$

$$= \min\left\{F_{2n-1}, F_{2n+1} - 2F_{2n} + F_{2n-1}\right\} = \min\left\{F_{2n-1}, F_{2n-3}\right\} = F_{2n-3}$$

W29. Solution by the proposer. Let P be a point in $\triangle ABC$ with barycentric coordinates

$$(p_a, p_b, p_c) = (x, y, z)$$

Let $R_a := PA$, $R_b = PB$, $R_c := PC$ and A_p, B_p, C_p be foots of perpendiculars from P to sides BC, CA, AB respectively. Also, we denote via

$$d_a := PA_p, d_b := PB_p, d_c := PC_p$$

and

$$a_p := B_pC_p, b_p := C_pA_p, c_p := A_pB_p$$

(side lengths of pedal triangle $A_pB_pC_p$). Let $F := [ABC]$ and

$$F_a := [PBC], F_b := [PCA], F_c := [PAB].$$

Then

$$F_a = \frac{ad_a}{2}, F_b = \frac{bd_b}{2}, F_c = \frac{cd_c}{2}$$