Minimization in Fibonacci Domain.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA

For any fixed natural n let $D_n := \{(x,y) \mid x,y \in \mathbb{Z}$ and $F_{n+1}x - F_n y = 1\}$, where F_n is n –th Fibonacci number. Find:

a) $\min\limits_{(x,y)\in D_n} |x+y|$;

b) $\min\limits_{(x,y)\in D_n} (|x|+|y|)$;

c) $\min\limits_{(x,y)\in D_n} (x^2+y^2)$.

Solution.

Since $F_{n+1}\cdot F_{n-1} - F_n^2 = (-1)^n$ (Cassini identity) then for $(x_0,y_0) = ((-1)^n F_{n-1}, (-1)^n F_n)$ we have $F_{n+1}x_0 - F_n y_0 = F_{n+1}\cdot(-1)^n F_{n-1} - F_n\cdot(-1)^n F_n = (-1)^n(F_{n+1}\cdot F_{n-1} - F_n^2) = 1$ and, therefore,

$F_{n+1}x - F_n y = 1 \Leftrightarrow F_{n+1}x - F_n y = F_{n+1}\cdot(-1)^n F_{n-1} - F_n\cdot(-1)^n F_n \Leftrightarrow F_{n+1}(x-(-1)^n F_{n-1}) =$
$F_n(y-(-1)^n F_n) \Leftrightarrow \begin{cases} x-(-1)^n F_{n-1} = tF_n \\ y-(-1)^n F_n = tF_{n+1} \end{cases} \Leftrightarrow \begin{cases} x = tF_n + (-1)^n F_{n-1} \\ y = tF_{n+1} + (-1)^n F_n \end{cases}, t \in \mathbb{Z}.$

Thus, $D_n = \{(x,y) \mid x = tF_n + (-1)^n F_{n-1}, y = tF_{n+1} + (-1)^n F_n, t \in \mathbb{Z}\}$

a) $\min\limits_{(x,y)\in D_n} |x+y| = \min\limits_{t\in\mathbb{Z}} |t(F_n+F_{n+1}) + (-1)^n(F_{n-1}+F_n)| = \min\limits_{t\in\mathbb{Z}} \varphi(t)$,

where $\varphi(t) := |tF_{n+2} + (-1)^n F_{n+1}|$.

Since $tF_{n+2} + (-1)^n F_{n+1} = 0 \Leftrightarrow t = t_* := \dfrac{(-1)^{n+1}F_{n+1}}{F_{n+2}}$ and $|t_*| = \dfrac{F_{n+1}}{F_{n+2}} < 1$ then integer t which minimize $\varphi(t)$ must be among closest to t_* integers, that is .

Thus, $\min\limits_{t\in\mathbb{Z}} \varphi(t) = \min\limits_{t\in\{-1,0,1\}} \varphi(t) = \min\{\varphi(0),\varphi(1),\varphi(-1)\} =$
$\min\{F_{n+1}, |F_{n+2}+(-1)^n F_{n+1}|, |-F_{n+2}+(-1)^n F_{n+1}|\}$.

If n is odd then $\min\{F_{n+1}, |F_{n+2}+(-1)^n F_{n+1}|, |-F_{n+2}+(-1)^n F_{n+1}|\} =$
$\min\{F_{n+1}, F_{n+2}-F_{n+1}, F_{n+2}+F_{n+1}\} = \min\{F_{n+1}, F_n, F_{n+3}\} = F_n = \varphi(1)$;

If n is even then $\min\{F_{n+1}, F_{n+2}+F_{n+1}, |-F_{n+2}+F_{n+1}|\} = \min\{F_{n+1}, F_{n+3}, F_n\} = F_n = \varphi(-1)$

So, $\min\limits_{(x,y)\in D_n} |x+y| = F_n$.

b) $|x|+|y| = |tF_n + (-1)^n F_{n-1}| + |tF_{n+1} + (-1)^n F_n|$.

By replacing t with $(-1)^{n-1}t$ we obtain $|x|+|y| = |tF_n - F_{n-1}| + |tF_{n+1} - F_n| =$
$\max\{|tF_n - F_{n-1} + tF_{n+1} - F_n|, |tF_n - F_{n-1} - tF_{n+1} + F_n|\} =$
$\max\{|t(F_n+F_{n+1}) - (F_{n-1}+F_n)|, |t(F_n - F_{n+1}) + F_n - F_{n-1}|\} =$
$\max\{|tF_{n+2} - F_{n+1}|, |-F_{n-1}t + F_{n-2}|\} = \max\{|tF_{n+2} - F_{n+1}|, |F_{n-1}t - F_{n-2}|\}$.

Lemma.

Let $p,q > 0, a,b \in \mathbb{R}$. Then $\min\limits_{x\in\mathbb{R}}(\max\{|px-a|, |qx-b|\}) = \dfrac{|aq-bp|}{p+q}$ and attained if $x = \dfrac{a+b}{p+q}$.

Proof.

First note that $\min\limits_{x\in\mathbb{R}}(\max\{|px-a|, |qx-b|\}) = \min t$ where
t provide solvability of inequality $t \geq \max\{|px-a|, |qx-b|\} \Leftrightarrow$

$$\begin{cases} t \geq |px-a| \\ t \geq |qx-b| \end{cases} \Leftrightarrow \begin{cases} -t \leq px-a \leq t \\ -t \leq qx-b \leq t \end{cases} \Leftrightarrow \begin{cases} \dfrac{a-t}{p} \leq x \leq \dfrac{t+a}{p} \\ \dfrac{b-t}{q} \leq x \leq \dfrac{t+b}{q} \end{cases} \Leftrightarrow$$

(**1**) $\max\left\{\dfrac{a-t}{p}, \dfrac{b-t}{q}\right\} \leq x \leq \min\left\{\dfrac{t+a}{p}, \dfrac{t+b}{q}\right\}$.

Condition of solvability is $\begin{cases} \dfrac{a-t}{p} \leq \dfrac{t+b}{q} \\ \dfrac{b-t}{q} \leq \dfrac{t+a}{p} \end{cases} \Leftrightarrow \begin{cases} q(a-t) \leq p(t+b) \\ p(b-t) \leq q(t+a) \end{cases} \Leftrightarrow$

$$\begin{cases} q(a-t) \leq p(t+b) \\ p(b-t) \leq q(t+a) \end{cases} \Leftrightarrow \begin{cases} \dfrac{aq-pb}{p+q} \leq t \\ \dfrac{pb-aq}{p+q} \leq t \end{cases} \Leftrightarrow \frac{|aq-pb|}{p+q} \leq t.$$

Thus, $\min\limits_{x\in\mathbb{R}}\left(\max\left\{ |px-a|, |qx-b|\right\}\right) = \dfrac{|aq-pb|}{p+q}$.

Assuming WLOG that $aq \geq pb$, by substitution $t = \dfrac{aq-pb}{p+q}$ in inequality (**1**) we obtain $x = \dfrac{a+b}{p+q}$.

Indeed, $\dfrac{t+a}{p} = \dfrac{\dfrac{aq-pb}{p+q}+a}{p} = \dfrac{2aq+ap-bp}{p(p+q)}$, $\dfrac{t+b}{q} = \dfrac{\dfrac{aq-pb}{p+q}+b}{q} = \dfrac{a+b}{p+q}$

and $\dfrac{2aq+ap-bp}{p(p+q)} - \dfrac{a+b}{p+q} = \dfrac{2}{p} \cdot \dfrac{aq-bp}{p+q} \geq 0$. Hence, $x \leq \dfrac{a+b}{p+q}$.

Similarly we can obtain $\dfrac{a+b}{p+q} \leq x$. So, $x = \dfrac{a+b}{p+q}$.

Corollary.

Let $p > q$ then $\min\limits_{x\in\mathbb{R}}\left(|px-a|+|qx-b|\right) = \dfrac{|aq-bp|}{p}$

Proof.

Since $|px-a|+|qx-b| = \max\{|(p+q)x-(a+b)|,|(p-q)x-(a-b)|\}$

then by **Lemma** $|px-a|+|qx-b| \geq \dfrac{|(a+b)(p-q)-(a-b)(p+q)|}{(p+q)+(p-q)} = \dfrac{|aq-bp|}{p}$

and equality occurs iff $x = \dfrac{(a+b)+(a-b)}{(p+q)+(p-q)} = \dfrac{2a}{2p} = \dfrac{a}{p}$.

So, $\min\limits_{x\in\mathbb{R}}\left(|px-a|+|qx-b|\right) = \dfrac{|aq-bp|}{p}$ and attained if $x = \dfrac{a}{p}$

Application.

Since $|x|+|y| = |tF_n - F_{n-1}| + |tF_{n+1} - F_n|$ and $F_{n+1} > F_n$ then by **Corollary** for real t minimum of $|tF_n - F_{n-1}| + |tF_{n+1} - F_n|$ attained if $t_* = \dfrac{F_n}{F_{n+1}} < 1$ and closest to t_* integer values of t are $t = 1$ and $t = 0$. Therefore, $\min\limits_{t\in\mathbb{Z}}|tF_n - F_{n-1}| + |tF_{n+1} - F_n| =$

$\min\{|0 \cdot F_n - F_{n-1}| + |0 \cdot F_{n+1} - F_n|, |1 \cdot F_n - F_{n-1}| + |1 \cdot F_{n+1} - F_n|\} =$

$\min\{F_{n-1} + F_n, F_n - F_{n-1} + F_{n+1} - F_n\} = \min\{F_{n+1}, F_n\} = F_n$.

c) $x^2+y^2 = (tF_n - F_{n-1})^2 + (tF_{n+1} - F_n)^2 = (F_n^2 + F_{n+1}^2)t^2 - 2F_n(F_{n-1}+F_{n+1})t + (F_n^2 + F_{n-1}^2)$.

Since $F_{n-1}^2 + F_n^2 = F_{2n-1}$ and $F_n\left(F_{n+1} + F_{n-1}\right) = F_{2n}$ then $x^2+y^2 = F_{2n+1}t^2 - 2F_{2n}t + F_{2n-1}$.

Quadratic trinomial $F_{2n+1}t^2 - 2F_{2n}t + F_{2n-1}$ attain minimum in $\mathbb{R}$ when $t = \dfrac{F_{2n}}{F_{2n+1}} < 1$ and, therefore,

minimum in $\mathbb{Z}$ can be attained in one of two integer points closest to $\frac{F_{2n}}{F_{2n+1}}$, that is when $t = 0$ or $t = 1$.

Thus, $\min(x^2 + y^2) = \min\{F_{2n+1} \cdot 0^2 - 2F_{2n} \cdot 0 + F_{2n-1}, F_{2n+1} \cdot 1^2 - 2F_{2n} \cdot 1 + F_{2n-1}\} = \min\{F_{2n-1}, F_{2n+1} - 2F_{2n} + F_{2n-1}\} = \min\{F_{2n-1}, F_{2n-3}\} = F_{2n-3}$.