W2. Let F_n be the n^{th} Fibonacci number defined by $F_0 = 0,\ F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Prove that $\dfrac{F_{2n+1} + F_nF_{n+1} + 1}{F_{n+2} + \sum\limits_{1\leq i<j\leq n} F_iF_j}$ is an integer number and determine its value.

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Solution by Arkady Alt, **San Jose**, **California**, **USA**.

1. Noting that $\sum\limits_{k=1}^{n-1} F_k = \sum\limits_{k=1}^{n-1}(F_{k+2} - F_{k+1}) = F_{n+1} - F_2 = F_{n+1} - 1$ and

$\sum\limits_{i=1}^{n-1} F_iF_{i+2} = \sum\limits_{i=1}^{n-1} F_i(F_{i+1} + F_i), \forall n \in \mathbb{N}\diagdown\{1\}$ we obtain

$$\sum_{1\leq i<j\leq n} F_iF_j = \sum_{i=1}^{n-1} F_i \sum_{j=i+1}^{n} F_j = \sum_{i=1}^{n-1} F_i \sum_{j=i+1}^{n} (F_{j+2} - F_{j+1}) = \sum_{i=1}^{n-1} F_i(F_{n+2} - F_{i+2}) =$$

$$F_{n+2}\sum_{i=1}^{n-1} F_i - \sum_{i=1}^{n-1} F_iF_{i+2} = F_{n+2}(F_{n+1} - 1) - \sum_{i=1}^{n-1} F_iF_{i+1} - \sum_{i=1}^{n-1} F_i^2 =$$

$$F_{n+2}F_{n+1} - F_{n+2} - \sum_{i=1}^{n-1} F_i^2 - \sum_{i=1}^{n-1} F_{i-1}F_i.$$

Since $\sum\limits_{k=m}^{n} F_k = \sum\limits_{k=m}^{n}(F_{k+2} - F_{k+1}) = F_{n+2} - F_{m+1}, m \leq n,\ F_nF_{n+2} = F_{n+1}^2 + (-1)^{n+1}$,

and $F_n^2 = F_n(F_{n+1} - F_{n-1})$ then $\sum\limits_{1\leq i<j\leq n} F_iF_j = \sum\limits_{i=1}^{n-1} F_i \sum\limits_{j=i+1}^{n} F_j = \sum\limits_{i=1}^{n-1} F_i \sum\limits_{j=i+1}^{n} (F_{j+2} - F_{j+1}) =$

$$\sum_{i=1}^{n-1} F_i(F_{n+2} - F_{i+2}) = F_{n+2}\sum_{i=1}^{n-1} F_i - \sum_{i=1}^{n-1} F_iF_{i+2} = F_{n+2}(F_{n+1} - 1) - \sum_{i=1}^{n-1}\left(F_{i+1}^2 + (-1)^{i+1}\right) =$$

$$F_{n+1}F_{n+2} - F_{n+2} - \sum_{i=1}^{n-1} F_{i+1}(F_{i+2} - F_i) + \sum_{i=1}^{n-1}(-1)^i =$$

$$F_{n+1}F_{n+2} - F_{n+2} - \sum_{i=1}^{n-1}(F_{i+1}F_{i+2} - F_iF_{i+1}) + \frac{(-1)^{n-1} - 1}{2} =$$

$$F_{n+1}F_{n+2} - F_{n+2} - F_nF_{n+1} + 1 + \frac{(-1)^{n-1} - 1}{2} = F_{n+1}F_{n+2} - F_{n+2} - F_nF_{n+1} + \frac{(-1)^{n-1} + 1}{2} =$$

$$F_{n+1}(F_{n+2} - F_n) - F_{n+2} + \frac{(-1)^{n-1} + 1}{2} = F_{n+1}^2 - F_{n+2} + \frac{(-1)^{n-1} + 1}{2}.$$

Thus, $\sum\limits_{1\leq i<j\leq n} F_iF_j = F_{n+1}^2 - F_{n+2} + \dfrac{(-1)^{n-1} + 1}{2} \Leftrightarrow F_{n+2} + \sum\limits_{1\leq i<j\leq n} F_iF_j = F_{n+1}^2 + \dfrac{(-1)^{n-1} + 1}{2}$.

Noting that $F_{2n+1} = F_{n+1}^2 + F_n^2$ we obtain $\dfrac{F_{2n+1} + F_nF_{n+1} + 1}{F_{n+2} + \sum\limits_{1\leq i<j\leq n} F_iF_j} =$

$$\frac{F_{n+1}^2 + F_n^2 + F_nF_{n+1} + 1}{F_{n+1}^2 + \dfrac{(-1)^{n-1} + 1}{2}} = \frac{F_{n+1}^2 + F_n(F_{n+1} + F_n) + 1}{F_{n+1}^2 + \dfrac{(-1)^{n-1} + 1}{2}} = \frac{F_{n+1}^2 + F_nF_{n+2} + 1}{F_{n+1}^2 + \dfrac{(-1)^{n-1} + 1}{2}} =$$

$$\frac{F_{n+1}^2 + F_{n+1}^2 + (-1)^{n+1} + 1}{F_{n+1}^2 + \dfrac{(-1)^{n-1} + 1}{2}} = \frac{2F_{n+1}^2 + (-1)^{n+1} + 1}{F_{n+1}^2 + \dfrac{(-1)^{n-1} + 1}{2}} = 2.$$