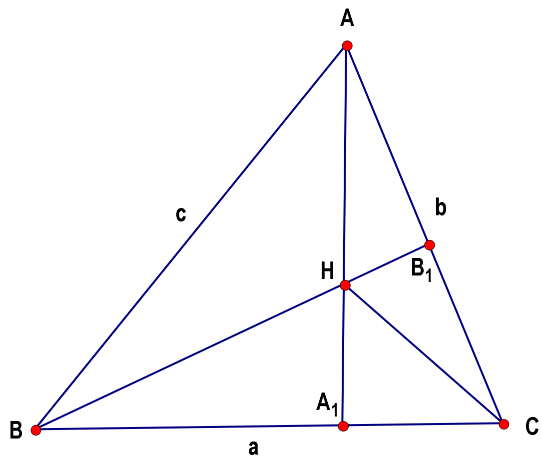
Problem proposed by Arkady Alt, San Jose, California, USA.
Maximum area of ortho-quadrilateral in acute triangle.
Let F be area of a acute triangle $\triangle ABC$ with bigger angle in C and let AA_1, BB_1 be altitudes from vertices A and B, respectively. Find among such triangles the triangle with greatest value of area of ortho-quadrilateral A_1CB_1H.



Solution.
Let $F_q := [HB_1CA_1]$ and $\alpha := \angle A, \beta := \angle B, \gamma := \angle C$. Since in fact we should find greatest value of ratio $\dfrac{F_q}{F}$ which is invariant of similarity then we can consider triangle with sides $BC = \sin\alpha, CA = \sin\beta, AB = \sin\gamma$ i.e. triangle with circumradius $R = \dfrac{1}{2}$.
Then $F = \dfrac{\sin\alpha\sin\beta\sin\gamma}{2}$. Also in such triangle $BA_1 = \sin\gamma\cos\beta, HA_1 = BA_1\cot\gamma = \cos\beta\cos\gamma, A_1C = \sin\beta\cos\gamma, AB_1 = \sin\gamma\cos\alpha, HB_1 = AB_1\cot\gamma = \cos\alpha\cos\gamma,$
$CB_1 = \sin\alpha\cos\gamma$. Hence, $[HB_1CA_1] = [HA_1C] + [HB_1C] = \dfrac{HA_1 \cdot A_1C}{2} + \dfrac{HB_1 \cdot B_1C}{2} = \dfrac{1}{2}(\cos\beta\cos\gamma \cdot \sin\beta\cos\gamma + \cos\alpha\cos\gamma \cdot \sin\alpha\cos\gamma) =$
$\dfrac{\cos^2\gamma}{2}(\cos\beta\sin\beta + \cos\alpha\sin\alpha) = \dfrac{\cos^2\gamma}{4}(\sin 2\alpha + \sin 2\beta) =$
$\dfrac{\cos^2\gamma\sin(\alpha+\beta)\cos(\alpha-\beta)}{2} = \dfrac{\cos^2\gamma\sin\gamma\cos(\alpha-\beta)}{2}$.
Thus we have $\dfrac{F_q}{F} = \dfrac{\cos^2\gamma\sin\gamma\cos(\alpha-\beta)}{\sin\alpha\sin\beta\sin\gamma} = \dfrac{\cos^2\gamma\cos(\alpha-\beta)}{\sin\alpha\sin\beta} =$
$\dfrac{2\cos^2\gamma\cos(\alpha-\beta)}{\cos(\alpha-\beta)-\cos(\alpha+\beta)} = \dfrac{2\cos^2\gamma\cos(\alpha-\beta)}{\cos(\alpha-\beta)+\cos(\alpha+\beta)} =$
$\dfrac{2\cos^2\gamma\cos(\alpha-\beta)+2\cos^3\gamma-2\cos^3\gamma}{\cos(\alpha-\beta)+\cos\gamma} = 2\cos^2\gamma - \dfrac{2\cos^3\gamma}{\cos(\alpha-\beta)+\cos\gamma} \le$
$2\cos^2\gamma\left(1 - \dfrac{\cos\gamma}{1+\cos\gamma}\right) = \dfrac{2\cos^2\gamma}{1+\cos\gamma}$.
Since $\gamma \ge \alpha, \beta$ then $\gamma \ge \dfrac{\pi}{3} \Leftrightarrow \dfrac{1}{2} \le \cos\gamma < 1 \Leftrightarrow 1 < \dfrac{1}{\cos\gamma} \le 2$ and,

therefore, $\dfrac{\cos^2\gamma}{1+\cos\gamma} = \dfrac{1}{\dfrac{1}{\cos^2\gamma} + \dfrac{1}{\cos\gamma}} \le \dfrac{1}{4+2} = \dfrac{1}{6}$

with condition of equality $\gamma = \dfrac{\pi}{3}$.

Thus $\max \dfrac{F_q}{F} = \dfrac{1}{3}$ and can be attained iff $\alpha = \beta$ and $\gamma = \dfrac{\pi}{3}$ i.e. triangle is equilateral.