Marco Bertolini

**W18. Solution by the proposer.** Since $D$ is symmetrical then $D = D_< \cup D_>$, where $D_< := \{(x, y) \mid x, y \in D \text{ and } x < y\}$ and $D_> := \{(x, y) \mid x, y \in D \text{ and } x > y\}$.
Let $f(x) := x^{\frac{1}{x}}, x > 0$. Since

$$f'(x) = \frac{f(x)}{x^2}(1 - \ln x)$$

then $f(x)$ strictly increasing on $(0, e]$ and strictly decreasing on $[0, \infty)$ with $\max\limits_{x>0} f(x) = f(e) = e^{\frac{1}{e}}$.
Therefore, noting that

$$x^y = y^x \iff x^{\frac{1}{x}} = y^{\frac{1}{y}} \iff f(x) = f(y),$$

we can conclude that $(x, y) \in D_< \implies x < e < y$,and $(x, y) \in D_> \implies y < e < x$. (otherwise if $x, y$ both belong to $(0, e)$ or $(e, \infty)$ then, due to monotonicity $f(x)$ equality $f(x) = f(y)$ yields $x = y$,that is the contradiction.
And, also, if $x = e$ then $y > e \implies f(e) = f(y) < f(e)$).
Also note that if $(x, y) \in D_<$ then $x > 1$.Indeed, since $x^y = y^x \iff y = x \log_x y$, then supposition $x < 1$ implies $0 < \dfrac{y}{x} = \log_x y < 0$, i.e. contradiction.
Hence $(x, y) \in D_< \implies x \in (1, e),\ \ y \in (e, \infty)$ and then

$$(x, y) \in D \implies x, y \in (1, e) \cup (e, \infty)$$

Let $t := \log_x y - 1$.Then $\log_x y = t + 1 \iff y = x^{t+1}$ and $y = x \log_x y \iff y = x(t + 1)$.
Hence $y = x^{t+1}$, and, therefore,

$$x^{t+1} = x(t + 1) \iff x^t = t + 1 \iff x = (t + 1)^{\frac{1}{t}} \implies y = (t + 1)^{1+\frac{1}{t}},$$

where $t > 0$ (since $1 < x < y$)
Thus

$$, D_< = \left\{\left((t + 1)^{\frac{1}{t}}, (t + 1)^{1+\frac{1}{t}}\right) \mid t \in (0, \infty)\right\}$$

is set of all non-trivial solution of equation $x^y = y^x$, satisfied $x < y$.

Since $t = \log_x y - 1$ then

$$(x,y) \in D_> \implies 1 < y < e < x \implies 1 < y < x \iff$$

$$\iff \log_x 1 - 1 < \log_x y - 1 < \log_x x - 1 \iff -1 < t < 0$$

Therefore,

$$D_> = \left\{ \left( (t+1)^{\frac{1}{t}}, (t+1)^{1+\frac{1}{t}} \right) \mid t \in (-1, 0) \right\}$$

and

$$D = \left\{ \left( (t+1)^{\frac{1}{t}}, (t+1)^{1+\frac{1}{t}} \right) \mid t \in (-1, 0) \cup (0, \infty) \right\}.$$

Note that $\lim\limits_{t\to 0} x = \lim\limits_{t\to 0} (t+1)^{\frac{1}{t}} = e$ and $\lim\limits_{t\to 0} y = \lim\limits_{t\to 0} (t+1)^{1+\frac{1}{t}} = e$.

Since for $(x,y) \in D_<$ we have $x = (1+t)^{\frac{1}{t}}$ and $y = (1+t)^{1+\frac{1}{t}}$ then

$$x^\alpha y^\beta = (1+t)^{\frac{\alpha}{t}} (1+t)^{\beta + \frac{\beta}{t}} = (1+t)^{\frac{\alpha+\beta}{t}+\beta} = \left( (1+t)^{\frac{1}{t}+\frac{\beta}{\alpha+\beta}} \right)^{\alpha+\beta}.$$

Since $\alpha \le \beta$ then $\dfrac{\beta}{\alpha+\beta} \ge \dfrac{\beta}{\beta+\beta} = \dfrac{1}{2}$ and, therefore,

$$x^\alpha y^\beta \ge \left( (1+t)^{\frac{1}{t}+\frac{1}{2}} \right)^{\alpha+\beta} > e^{\alpha+\beta}$$

(because $(1+t)^{\frac{1}{t}+\frac{1}{2}}$ is increasing* on $(0,\infty)$ and $\lim\limits_{t\to 0} (1+t)^{\frac{1}{t}+\frac{1}{2}} = e$).

Also we have

$$\lim_{t\to 0} x^\alpha y^\beta = \lim_{t\to 0} (1+t)^{\frac{\alpha+\beta}{t}+\beta} = e^{\alpha+\beta}.$$

Thus,

$$\inf_{(x,y)\in D} x^\alpha y^\beta = \inf_{(x,y)\in D_<} x^\alpha y^\beta = e^{\alpha+\beta}.$$

*

$$\left( \left(1+\frac{2}{t}\right) \ln(t+1) \right)' = \left(-\frac{2}{t^2}\right) \ln(t+1) + \left(1+\frac{2}{t}\right) \cdot \frac{1}{t+1} =$$

$$= \frac{1}{t^2} \left( \frac{t(t+2)}{t+1} - 2\ln(t+1) \right).$$

We will prove that $\frac{t(t+2)}{t+1}-2\ln(t+1)>0$ for any $t>0$.
Indeed,

$$\left(\frac{t(t+2)}{t+1}-2\ln(t+1)\right)'=\left(t+1-\frac{1}{t+1}-2\ln(t+1)\right)'=$$

$$=1+\frac{1}{(t+1)^2}-\frac{2}{t+1}=\frac{t^2+2t+2-2t-2}{(t+1)^2}=\frac{t^2}{(t+1)^2}>0.$$

**Second solution.**

$$\inf_{(x,y)\in D} x^\alpha y^\beta=4^\alpha 2^\beta=2^{2\alpha+\beta}$$

$\alpha\le\beta,\ \inf_{(x,y)\in D} x^\alpha y^\beta=1$

Ayodeji Salau

**W19. Solution by the proposer.** Let $S_p(n):=1^p+2^p+...+n^p$. First we will prove

**Lemma.** $S_p(n)\ge\frac{n^p(n+1)}{p+1}$ for any $p,n\in\mathbb{N}$.

*Proof.* First note that for $p=1$ we have

$$S_1(n)=\frac{n(n+1)}{2}\ge\frac{n^1(n+1)}{1+1}.$$

Let $p\ge\mathbb{N}\setminus\{1\}$ be any.

For $n=1$ we have $S_p(1)=1$ and $\frac{1^p(1+1)}{p+1}=\frac{2}{p+1}<1$.

Step of Math Induction.

We will prove that

$$S_p(n+1)-S_p(n)\ge\frac{(n+1)^p(n+2)}{p+1}-\frac{n^p(n+1)}{p+1}\iff$$

$$\iff (n+1)^p\ge\frac{(n+1)^p(n+2)-n^p(n+1)}{p+1}\iff$$

$$\iff (p+1)(n+1)^{p-1}\ge(n+1)^{p-1}(n+2)-n^p\iff$$

$$\iff (p-1)(n+1)^{p-1}\ge n(n+1)^{p-1}-n^p\iff$$

$$\iff (p-1)(n+1)^{p-1} + n^p - n(n+1)^{p-1} \geq 0.$$

We have

$$(p-1)(n+1)^{p-1} + n^p - n(n+1)^{p-1} =$$

$$= (p-1)\sum_{k=0}^{p-1}\binom{p-1}{k}n^k + n^p - n\sum_{k=0}^{p-1}\binom{p-1}{k}n^k =$$

$$= (p-1)\sum_{k=0}^{p-1}\binom{p-1}{k}n^k - n\sum_{k=0}^{p-2}\binom{p-1}{k}n^k =$$

$$= p-1+(p-1)\sum_{k=1}^{p-1}\binom{p-1}{k}n^k - n\sum_{k=0}^{p-2}\binom{p-1}{k}n^k =$$

$$= p-1+(p-1)\sum_{k=0}^{p-2}\binom{p-1}{k+1}n^{k+1} - \sum_{k=0}^{p-2}\binom{p-1}{k}n^{k+1} =$$

$$= p-1+\sum_{k=0}^{p-2}\left((p-1)\binom{p-1}{k+1} - \binom{p-1}{k}\right)n^{k+1}.$$

Since for any $p \in \mathbb{N}$ and $k = 0, 1, ..., p-1$ we have

$$p\binom{p}{k+1} - \binom{p}{k} = \frac{p \cdot p!}{(k+1)!\,(p-k-1)!} - \frac{p!}{k!\,(p-k)!} =$$

$$= \frac{p!}{k!\,(p-k)!}\left(\frac{p(p-k)}{k+1} - 1\right) \geq 0$$

(because $k \leq p-1 \implies \dfrac{p(p-k)}{k+1} \geq \dfrac{p(p-(p-1))}{(p-1)+1} = 1$) then

$$(p-1)(n+1)^{p-1} + n^p - n(n+1)^{p-1} =$$

$$= p-1+\sum_{k=0}^{p-2}\left((p-1)\binom{p-1}{k+1} - \binom{p-1}{k}\right)n^{k+1} \geq 0$$

Since

$$\frac{n^{p-1}}{S_p(n)} \geq \frac{p+1}{n(n+1)}$$

then

$$\prod_{i=k}^{n}\left(1+\frac{n^{p-1}}{S_p(n)}\right)\leq\prod_{k=2}^{n}\left(1+\frac{p+1}{k(k+1)}\right)$$

and by AM-GM Inequality

$$\prod_{k=2}^{n}\left(1+\frac{p+1}{k(k+1)}\right)\leq\left(\frac{\sum\limits_{k=2}^{n}\left(1+\frac{p+1}{k(k+1)}\right)}{n-1}\right)^{n-1}=$$

$$=\left(\frac{n-1+(p+1)\sum\limits_{k=2}^{n}\frac{1}{k(k+1)}}{n-1}\right)^{n-1}<$$

$$<\left(\frac{n-1+\frac{p+1}{2}}{n-1}\right)^{n-1}=\left(1+\frac{\frac{p+1}{2}}{n-1}\right)^{n-1}<e^{(p-1)/2}.$$

**W20. Solution by the proposer.** Since recurrence $a_{n+1}=\dfrac{a_n}{1+a_n^p}$ can be rewritten in the form $a_{n+1}^p=\dfrac{a_n^p}{(1+a_n^\alpha)^p}$, then denoting $a_n^p$ via $b_n$ we obtain recurrence

$$b_{n+1}=\frac{b_n}{(1+b_n)^p} \tag{1}$$

with initial condition $b_1=a^p$.

For convenience we set $a:=(b^q-1)^q$, where $b>1$ and $q:=\dfrac{1}{p}$.Then $b_1:=b^q-1$.

**Lemma.** There are two constants $l$ and $r$ such that for any natural $n$ holds inequality

$$q(n-lh_n-1)<\frac{1}{b_n}<q(n+r), \tag{L}$$

where $h_n = 1 + \frac{1}{2} + ... + \frac{1}{n}$.

*Proof.* From the recurrence (1) obviously follows, that $b_n$ decreasing in $\mathbb{N}$. In particularly this yields $b_n \leq b_1 = b^q - 1$.
Denoting $h(x) := \frac{(1+x)^p - 1}{x}$, we can rewrite recurrence **(1)** in the form

$$\frac{1}{b_{n+1}} - \frac{1}{b_n} = h(b_n) \cdot \tag{2}$$

Recall for further Bernoulli Inequalities

$$(1+t)^\alpha > 1 + \alpha t \ , \quad \alpha > 1 \tag{B1}$$

$t > -1$
and

$$(1+t)^\alpha < 1 + \alpha t, \alpha \in (0,1) \ , t > -1 \tag{B2}$$

And also note that $h(x)$ decreasing in $(0, \infty)$

$$h'(x) = \frac{(1+x)^{p-1}\left((1+x)^{p-1} - 1 - (1-p)x\right)}{x^2} < 0$$

because by **(B2)** $(1+x)^{p-1} < 1 + (1-p)x$
Let $m$ natural number such that $mp > 1$ ($m$ exist due to Archimed's axiom). Applying **(B1)** to $t = x > 0$ and $\alpha = mp$ we obtain $(1+x)^{mp} > 1 + mpx$. Hereof

$$(1+x)^p = ((1+x)^{mp})^{\frac{1}{m}} > (1+mpx)^{\frac{1}{m}}$$

and then

$$h(x) = \frac{(1+x)^p - 1}{x} > \frac{(1+mpx)^{\frac{1}{m}} - 1}{x} = \frac{mp}{\sum\limits_{k=0}^{m-1} (1+mpx)^{\frac{k}{m}}}.$$

Since for $k = 1, 2, ..., m-1$ by inequality (B2) we have

$$(1+mpx)^{\frac{k}{m}} < 1 + mpx \cdot \tfrac{k}{m} = 1 + kpx$$

then

$$h(x) > \frac{mp}{\sum\limits_{k=0}^{m-1}(1+kpx)} = \frac{mp}{m+px\cdot\dfrac{m(m-1)}{2}} = \frac{2p}{2+px(m-1)}.$$

From the other hand, applying (B2) to $t = x$ and $\alpha = p$ we obtain

$$h(x) < \frac{1+px-1}{x} = p.$$

So, for any $x > 0$ holds inequality

$$\frac{2p}{2+px(m-1)} < h(x) < p$$

which together with identity **(2)** gives us for any $n \in \mathbb{N}$ inequality

$$\frac{2p}{2+p(m-1)b_n} < \frac{1}{b_{n+1}} - \frac{1}{b_n} < p. \quad (3)$$

Since $\dfrac{1}{b_{n+1}} - \dfrac{1}{b_n} = h(b_n)$ and $h(x)$ decreasing in $(0,\infty)$ then $h(b_n) \ge h(b_1) = \dfrac{b-1}{b^q-1}$ and $\dfrac{1}{b_{n+1}} - \dfrac{1}{b_n} \ge \dfrac{b-1}{b^q-1}$ for any natural $n$. Hence,

$$\frac{1}{b_{n+1}} - \frac{1}{b_1} = \sum_{k=1}^{n}\left(\frac{1}{b_{n+1}} - \frac{1}{b_n}\right) =$$

$$= \sum_{k=1}^{n} h(b_n) \ge n\cdot h(b_1) = \frac{n(b-1)}{b^q-1}$$

and

$$\frac{1}{b_{n+1}} = \frac{n(b-1)+1}{b^q-1}.$$

Since $b(n(b-1)+1) > (b-1)(n+1)$ then

$$\frac{n(b-1)+1}{b^q-1} > \frac{n+1}{c}$$

where $c := \dfrac{b(b^q-1)}{b-1}$. Thus $\dfrac{1}{b_{n+1}} > \dfrac{n+1}{c}$, $n \in \mathbb{N}$ and because $b_1 < \dfrac{c}{1} = \dfrac{b(b^q-1)}{b-1}$ then inequality $b_n < \dfrac{c}{n}$ holds for all natural $n$.

Since $b_n < \frac{c}{n}$ then

$$\frac{2p}{2+p(m-1)b_n} > \frac{2p}{2+p(m-1)\cdot\frac{c}{n}} = \frac{2pn}{2n+p(m-1)c} =$$

$$= p\left(1-\frac{p(m-1)c}{2n+p(m-1)c}\right) > p\left(1-\frac{p(m-1)c}{2n}\right).$$

Denoting $l := \frac{p(m-1)c}{2}$ and combined inequality

$$\frac{2p}{2+p(m-1)b_n} > p\left(1-\frac{p(m-1)c}{2n}\right)$$

with inequality (3) we obtain inequality

$$p\left(1-\frac{l}{n}\right) < \frac{1}{b_{n+1}} - \frac{1}{b_n} < p, \ n \in \mathbb{N} \tag{4}$$

Hence,

$$\sum_{k=1}^{n} p\left(1-\frac{l}{k}\right) < \sum_{k=1}^{n}\left(\frac{1}{b_{k+1}}-\frac{1}{b_k}\right) < \sum_{k=1}^{n} p \iff$$

$$\iff p(1-lh_n) < \frac{1}{b_{n+1}} - \frac{1}{b_1} < pn. \tag{5}$$

Since $\frac{1}{b_1} + pn = p(n+1+r)$ , where $r := \frac{1}{p(b^p-1)} - 1$, then from (5) and

$$p(n-l\cdot h_n) + \frac{1}{b_1} > p(n-l\cdot h_n) > p(n+1-l\cdot h_{n+1}-1)$$

follows inequality (L).
Dividing inequality (L) by $n$ we obtain

$$p\left(1-l\cdot\frac{h_n}{n}-\frac{1}{n}\right) < \frac{1}{nb_n} < p\left(1+\frac{r}{n}\right)$$

Since* $\lim_{n\to\infty}\frac{h_n}{n} = 0$ then

$$\lim_{n\to\infty} p\left(1-l\cdot\frac{h_n}{n}-\frac{1}{n}\right) = \lim_{n\to\infty} p\left(1+\frac{r}{n}\right) = p$$

and, therefore, $\lim_{n\to\infty}\frac{1}{nb_n} = p$ as well. Thus, $\lim_{n\to\infty} nb_n = q$ and we finally obtain that $\lim_{n\to\infty} n^q a_n = q^q$. So, $a_n \sim \left(\frac{q}{n}\right)^q$.

* From

$$\frac{h_n}{n} < \left( \frac{1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}}{n} \right)^{\frac{1}{2}}$$

and

$$1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2} < 1 + \frac{1}{1 \cdot 2} + \ldots + \frac{1}{(n-1)\,n} < 2$$

follows inequality $\frac{h_n}{n} < \sqrt{\frac{2}{n}}$, which immediately implies $\lim\limits_{n\to\infty} \frac{h_n}{n} = 0.$

**W21. Solution by the proposer.** Let $L$ be the proposed limit, and $L_1 = \lim\limits_{n\to\infty} \frac{1 + \frac{1}{3} + \cdots + \frac{1}{2n+1}}{\ln \sqrt{n}}$. Then, by the Stolz-Cesaro Lemma,

$$L_1 = \lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\ln\sqrt{n} - \ln\sqrt{n-1}} = \lim_{n\to\infty} \frac{\frac{2n}{2n+1}}{n \ln\left(\frac{n}{n-1}\right)} =$$

$$= \lim_{n\to\infty} \frac{1}{\ln\left(\frac{n}{n-1}\right)^n} = \frac{1}{\ln e} = 1$$

and so, the proposed limit is of the form $1^\infty$. Therefore

$$L = e^{\lim\limits_{n\to\infty} \left( \frac{1 + \frac{1}{3} + \cdots + \frac{1}{2n+1}}{\ln\sqrt{n}} - 1 \right) \ln\sqrt{n}} =$$

$$= e^{\lim\limits_{n\to\infty} \left( 1 + \frac{1}{3} + \cdots + \frac{1}{2n+1} - \ln\sqrt{n} \right)}.$$

Let $H_n$ denote the $n$-th harmonic number, that is $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Note that

$$\lim_{n\to\infty} \left( 1 + \frac{1}{3} + \cdots + \frac{1}{2n+1} - \ln\sqrt{n} \right) =$$

$$= \lim_{n\to\infty} H_{2n} - \frac{H_n}{2} - \ln n + \frac{\ln n}{2} = \lim_{n\to\infty} (H_{2n} - \ln n) - \lim_{n\to\infty} \left( \frac{H_n}{2} - \frac{\ln n}{2} \right) =$$