Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers such that $a_1 < a_2 < \ldots < a_n$ and $a_i \in (b_i, b_{i+1}), i = 1, 2, \ldots, n-1$. Let $F(x) := \dfrac{(x-b_1)(x-b_2)\ldots(x-b_n)}{(x-a_1)(x-a_3)\ldots(x-a_n)}$.

Prove that derivative of function $F(x)$ is negative everywhere where function is defined. (Or, prove that $F'(x) < 0$ for any $x \in Dom(F)$).

Solution.

Lemma.

$F(x)$ can be represented in form

$$F(x) = 1 + \sum_{k=1}^{n} \frac{c_k}{x-a_k},$$

where $c_k, k = 1, 2, \ldots, n$ are some positive real numbers.

Proof.

Let $F_k(x) := \dfrac{(x-b_1)(x-b_2)\ldots(x-b_k)}{(x-a_1)(x-a_3)\ldots(x-a_k)}, \ k \leq n.$

We will prove by Math. Induction that for any $k \leq n$ there are positive numbers $c_k(i), i = 1, \ldots, k$ such that $F_k(x) = 1 + \sum\limits_{i=1}^{k} \dfrac{c_k(i)}{x-a_i}$.

Let $d_k := a_k - b_k > 0, k = 1, 2, \ldots, n.$

Note that $F_1(x) = \dfrac{x-b_1}{x-a_1} = \dfrac{x-a_1+a_1-b_1}{x-a_1} = 1 + \dfrac{d_1}{x-a_1}$.

Since $\dfrac{x-b_{k+1}}{x-a_{k+1}} = 1 + \dfrac{d_{k+1}}{x-a_{k+1}}$ then in supposition $F_k(x) = 1 + \sum\limits_{i=1}^{k} \dfrac{c_k(i)}{x-a_i}$, where $c_k(i) > 0, i = 1, \ldots, k < n$ we obtain

$$F_{k+1}(x) = F_k(x) \cdot \frac{x-b_{k+1}}{x-a_{k+1}} = \left(1 + \sum_{i=1}^{k} \frac{c_k(i)}{x-a_i}\right)\left(1 + \frac{d_{k+1}}{x-a_{k+1}}\right) = 1 + \frac{d_{k+1}}{x-a_{k+1}} +$$
$$\sum_{i=1}^{k} \frac{c_k(i)}{x-a_i} + \sum_{i=1}^{k} \frac{d_{k+1}c_k(i)}{(x-a_i)(x-a_{k+1})} = 1 + \frac{d_{k+1}}{x-a_{k+1}} + \sum_{i=1}^{k} \frac{c_k(i)}{x-a_i} -$$
$$\sum_{i=1}^{k} \frac{d_{k+1}c_k(i)}{a_{k+1}-a_i}\left(\frac{1}{x-a_i} - \frac{1}{x-a_{k+1}}\right) = 1 + \frac{d_{k+1}}{x-a_{k+1}}\left(1 + \sum_{i=1}^{k} \frac{c_k(i)}{a_{k+1}-a_i}\right) +$$
$$\sum_{i=1}^{k} \frac{c_k(i)}{x-a_i}\left(1 - \frac{d_{k+1}}{a_{k+1}-a_i}\right) = 1 + \frac{d_{k+1}F_k(a_{k+1})}{x-a_{k+1}} + \sum_{i=1}^{k} \frac{c_k(i)}{x-a_i} \cdot \frac{b_{k+1}-a_i}{a_{k+1}-a_i}.$$

Since $F_k(a_{k+1}) > 0$ and $b_{k+1} - a_i = (b_{k+1} - a_k) + (a_k - a_i) > 0$ then $c_{k+1}(k+1) = d_{k+1}F_k(a_{k+1}) > 0, \ c_{k+1}(i) := \dfrac{(b_{k+1}-a_i)c_k(i)}{a_{k+1}-a_i} > 0, i = 1, 2, \ldots, k$ and $F_{k+1}(x) = 1 + \sum\limits_{i=1}^{k+1} \dfrac{c_{k+1}(i)}{x-a_i}$.

Since $F(x) = 1 + \sum\limits_{k=1}^{n} \dfrac{c_k}{x-a_k}$ and $c_k > 0, k = 1, 2, \ldots, n$ then $F'(x) = -\sum\limits_{k=1}^{n} \dfrac{c_k}{(x-a_k)^2} < 0$ for any $x \in Dom(F) = \mathbb{R} \setminus \{a_1, a_2, \ldots, a_n\}$.