W15. Let the sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$:

$$a_n = \sum_{k=1}^{n} \arctan \frac{1}{k^2-k+1} \quad \text{and} \quad \lim_{n\to\infty} \frac{b_{n+1}}{nb_n} = b \in \mathbb{R}_+^*.$$

Compute:

$$\lim_{n\to\infty}\left(\frac{\pi}{2} - a_n\right)\sqrt[n]{b_n}$$

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Solution by Arkady Alt, San Jose, California, USA.

Since $\arctan \frac{1}{k^2-k+1} = \arctan \frac{1}{k-1} - \arctan \frac{1}{k}$ (because

$\tan\left(\arctan \frac{1}{k-1} - \arctan \frac{1}{k}\right) = \dfrac{\frac{1}{k-1} - \frac{1}{k}}{1 + \frac{1}{k-1}\cdot\frac{1}{k}} = \dfrac{1}{k^2-k+1}$) then

$$a_n = \sum_{k=1}^{n} \arctan \frac{1}{k^2-k+1} = \sum_{k=1}^{n}\left(\arctan \frac{1}{k-1} - \arctan \frac{1}{k}\right) = \frac{\pi}{2} - \arctan \frac{1}{n}$$

and, therefore, $\lim_{n\to\infty}\left(\frac{\pi}{2} - a_n\right)\sqrt[n]{b_n} = \lim_{n\to\infty} \arctan \frac{1}{n}\sqrt[n]{b_n} =$

$$\lim_{n\to\infty}\left(n \arctan \frac{1}{n}\cdot\frac{\sqrt[n]{b_n}}{n}\right) = \lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}}.$$

Since $\lim_{n\to\infty}\left(\dfrac{b_{n+1}}{(n+1)^{n+1}} \div \dfrac{b_n}{n^n}\right) = \lim_{n\to\infty}\left(\dfrac{b_{n+1}}{b_n}\cdot\dfrac{n^n}{(n+1)^{n+1}}\right) =$

$\lim_{n\to\infty}\left(\dfrac{b_{n+1}}{nb_n}\cdot\dfrac{1}{\left(1+\frac{1}{n}\right)^{n+1}}\right) = e^{-1}\cdot b$ then by Multiplicative Stolz-Cesaro Theorem

$$\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}} = e^{-1}\cdot b.$$