W14. Prove that $\sqrt{1+\sqrt[3]{u}+\sqrt[3]{\overline{u}}}+\sqrt{\sqrt[3]{u}+\sqrt[3]{\overline{u}}}-\sqrt[3]{u}-\sqrt[3]{\overline{u}} \in \mathbb{N}$,
where $u=2+\frac{2}{3}\sqrt{\frac{11}{3}}, \overline{u}=2-\frac{2}{3}\sqrt{\frac{11}{3}}$.

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Let $x:=\sqrt[3]{u}+\sqrt[3]{\overline{u}}$. Since $u+\overline{u}=4, \sqrt[3]{u}\cdot\sqrt[3]{\overline{u}}=\sqrt[3]{4-\frac{4}{9}\cdot\frac{11}{3}}=\frac{4}{3}$ then
$x^3=u+\overline{u}+3\sqrt[3]{u}\cdot\sqrt[3]{\overline{u}}\cdot x=4+4x$ and, therefore,
$\sqrt{1+\sqrt[3]{u}+\sqrt[3]{\overline{u}}}+\sqrt{\sqrt[3]{u}+\sqrt[3]{\overline{u}}}-\sqrt[3]{u}-\sqrt[3]{\overline{u}}=\sqrt{1+x}+\sqrt{x}-x$,
where x is positive root of equation $x^3-4x-4=0$.
We have $x^3-4x-4=0 \overset{x>0}{\Longleftrightarrow} x^4=4x^2+4x=0 \Leftrightarrow x^2=2\sqrt{(1+x)x} \Leftrightarrow$
$1+2x+x^2=1+2x+2\sqrt{(1+x)x} \Leftrightarrow (1+x)^2=\left(\sqrt{1+x}+\sqrt{x}\right)^2 \Leftrightarrow$
$\sqrt{1+x}+\sqrt{x}=1+x \Leftrightarrow \sqrt{1+x}+\sqrt{x}-x=1$.
Thus, $\sqrt{1+\sqrt[3]{u}+\sqrt[3]{\overline{u}}}+\sqrt{\sqrt[3]{u}+\sqrt[3]{\overline{u}}}-\sqrt[3]{u}-\sqrt[3]{\overline{u}}=1$.