$$\lim_{n\to\infty} x_n = \delta + \sum_{j=1}^{k} c_j \lim_{n\to\infty} z_j^n = \delta.$$

W10. (Solution by the proposer.) First we will prove, using Math. Induction, inequality for $m = n$, namely inequality

$$\left(\frac{a^n + b^n + c^n}{a^{n-1} + b^{n-1} + c^{n-1}}\right)^{n+1} \geq \frac{a^{n+1} + b^{n+1} + c^{n+1}}{3}, \ n \geq 3. \tag{1}$$

1. *Base of Math. Induction.*
For $n = 3$ we have

$$\left(\frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2}\right)^4 \geq \frac{a^4 + b^4 + c^4}{3}. \tag{2}$$

Assume $a + b + c = 1$ (due to homogeneity of (2) and denote $p := ab + bc + ca, q := abc$. Since

$$a^2 + b^2 + c^2 = 1 - 2p, a^3 + b^3 + c^3 = 1 + 3q - 3p, a^4 + b^4 + c^4 =$$

$$= 1 + 4q - 4p + 2p^2$$

inequality (2) becomes

$$\left(\frac{1 + 3q - 3p}{1 - 2p}\right)^4 \geq \frac{1 + 4q - 4p + 2p^2}{3} \iff h(q) \geq \frac{(1 - 2p)^4}{3}$$

where

$$h(q) := \frac{(1 + 3q - 3p)^4}{1 + 4q - 4p + 2p^2}.$$

Noting that $p \leq \frac{1}{3}$ ($\iff ab + bc + ca \leq \frac{(a + b + c)^2}{3}$) and $9q \geq 4p - 1$($\iff \sum_{cyc} a(a - b)(a - c) \geq 0$ –Schure inequality in *1-p-q* notation) we can see that

$$h'(q) = \frac{12(1 + 3q - 3p)^3 (1 + 4q - 4p + 2p^2) - 4(1 + 3q - 3p)^4}{1 + 4q - 4p + 2p^2} =$$

$$= \frac{4(1 + 3q - 3p)^3 (6p^2 - 9p + 2 + 9q)}{1 + 4q - 4p + 2p^2} \geq 0$$

for $q \geq q_* := \dfrac{4p-1}{9}$ and $p \leq \dfrac{1}{3}$, because

$$6p^2 - 9p + 2 + 9q \geq 6p^2 - 9p + 2 + 4p - 1 = (1-3p)(1-2p).$$

Since $h(q)$ is increasing in $q \geq q_*$ then suffice to prove that

$$h(q_*) \geq \frac{(1-2p)^4}{3} \iff$$

$$\iff 3(1+3q_* - 3p)^4 - \left(1 + 4q_* - 4p + 2p^2\right)(1-2p)^4 \geq 0.$$

We have

$$3(1+3q_* - 3p)^4 - \left(1 + 4q_* - 4p + 2p^2\right)(1-2p)^4 =$$

$$= 3\left(1 + \frac{4p-1}{3} - 3p\right)^4 - (1-2p)^4\left(1 + \frac{4(4p-1)}{9} - 4p + 2p^2\right) =$$

$$= \frac{1}{27}(1-3p)\left(1 + 23p - 225p^2 + 677p^3 - 800p^4 + 288p^5\right).$$

Thus remains to prove $1 + 23p - 225p^2 + 677p^3 - 800p^4 + 288p^5 \geq 0$ for $p \in \left(0, \dfrac{1}{3}\right]$.

Let $t := 1 - 3p$, then $p = \dfrac{1-t}{3}$, $p \in \left(0, \dfrac{1}{3}\right] \iff t \in [0,1)$ and

$$1 + 23p - 225p^2 + 677p^3 - 800p^4 + 288p^5 =$$

$$= \frac{1}{81}\left(4 + 56t + 228t^2 + 209t^3 - 320t^4 - 96t^5\right) > 0$$

because $4 + 56t + 228t^2 + 209t^3 - 320t^4 - 96t^5 =$

$$4 + 56t + 21t^2 + 207t^2(1-t) + 416t^3(1-t) + 96t^4(1-t) \geq 4.$$

2. *Step of Math. Induction.*
Let

$$a_n := \left(\frac{a^n + b^n + c^n}{a^{n-1} + b^{n-1} + c^{n-1}}\right)^{n+1}$$

and

$$b_n := \frac{a^{n+1}+b^{n+1}+c^{n+1}}{2}.$$

Then

$$\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n} \iff \frac{\left(a^{n+1}+b^{n+1}+c^{n+1}\right)^{n+2}\left(a^{n-1}+b^{n-1}+c^{n-1}\right)^{n+1}}{\left(a^n+b^n+c^n\right)^{n+2}\left(a^n+b^n+c^n\right)^{n+1}} \geq$$

$$\geq \frac{a^{n+2}+b^{n+2}+c^{n+2}}{a^{n+1}+b^{n+1}+c^{n+1}} \iff$$

$$\iff \frac{\left(a^{n+1}+b^{n+1}+c^{n+1}\right)^{n+1}\left(a^{n-1}+b^{n-1}+c^{n-1}\right)^{n+1}}{\left(a^n+b^n+c^n\right)^{n+1}\left(a^n+b^n+c^n\right)^{n+1}} \geq$$

$$\geq \frac{\left(a^{n+2}+b^{n+2}+c^{n+2}\right)\left(a^n+b^n+c^n\right)}{\left(a^{n+1}+b^{n+1}+c^{n+1}\right)^2} \iff$$

$$\iff \left(\frac{\left(a^{n+1}+b^{n+1}+c^{n+1}\right)\left(a^{n-1}+b^{n-1}+c^{n-1}\right)}{\left(a^n+b^n+c^n\right)^2}\right)^{n+1} \geq$$

$$\geq \frac{\left(a^{n+2}+b^{n+2}+c^{n+2}\right)\left(a^n+b^n+c^n\right)}{\left(a^{n+1}+b^{n+1}+c^{n+1}\right)^2} \iff$$

$$\iff \left(1+\frac{\sum\left(a^{n-1}b^{n-1}(a-b)^2\right)}{\left(a^n+b^n+c^n\right)^2}\right)^{n+1} \geq 1+\frac{\sum\left(a^nb^n(a-b)^2\right)}{\left(a^{n+1}+b^{n+1}+c^{n+1}\right)^2}$$

and, by Bernoulli Inequality, we have

$$\left(1+\frac{\sum\left(a^{n-1}b^{n-1}(a-b)^2\right)}{\left(a^n+b^n+c^n\right)^2}\right)^{n+1} \geq 1+\frac{(n+1)\sum\left(a^{n-1}b^{n-1}(a-b)^2\right)}{\left(a^n+b^n+c^n\right)^2}.$$

Thus suffice to prove

$$\frac{(n+1)\sum\left(a^{n-1}b^{n-1}(a-b)^2\right)}{\left(a^n+b^n+c^n\right)^2} \geq \frac{\sum\left(a^nb^n(a-b)^2\right)}{\left(a^{n+1}+b^{n+1}+c^{n+1}\right)^2} \iff$$

$$\iff \sum_{cyc}(a-b)^2 a^{n-1}b^{n-1}\left(\frac{n+1}{(a^n+b^n+c^n)^2}-\frac{ab}{(a^{n+1}+b^{n+1}+c^{n+1})^2}\right)\geq 0.$$

Since $n\geq 3$ and $a^{n+1}+b^{n+1}+c^{n+1}\geq \dfrac{(a^n+b^n+c^n)(a+b+c)}{3}$ then

$$\frac{n+1}{(a^n+b^n+c^n)^2}-\frac{bc}{(a^{n+1}+b^{n+1}+c^{n+1})^2}\geq \frac{4}{(a^n+b^n+c^n)^2}-$$

$$-\frac{9bc}{(a^n+b^n+c^n)^2(a+b+c)^2}=\frac{4(a+b+c)^2-9bc}{(a^n+b^n+c^n)^2(a+b+c)^2}\geq 0$$

because

$$4(a+b+c)^2-9bc=4a^2+\left(4b^2-bc+4c^2\right)+8ab+8ca>0.$$

Now note, that for any $m\geq n\geq 1$ holds inequality

$$\left(\frac{a^m+b^m+c^m}{a^{m-1}+b^{m-1}+c^{m-1}}\right)^n\geq \frac{a^n+b^n+c^n}{3}. \tag{3}$$

Indeed, since $\dfrac{a^m+b^m+c^m}{a^{m-1}+b^{m-1}+c^{m-1}}$ is increasing in $m\in\mathbb{N}$

$$\frac{a^{m+1}+b^{m+1}+c^{m+1}}{a^m+b^m+c^m}\geq \frac{a^m+b^m+c^m}{a^{m-1}+b^{m-1}+c^{m-1}} \iff$$

$$\iff \left(a^{m+1}+b^{m+1}+c^{m+1}\right)\left(a^{m-1}+b^{m-1}+c^{m-1}\right)\geq (a^m+b^m+c^m)^2 \iff$$

$$\iff a^{m-1}b^{m-1}(a-b)^2+b^{m-1}c^{m-1}(b-c)^2+c^{m-1}a^{m-1}(c-a)^2\geq 0$$

then

$$m\geq n \implies \frac{a^m+b^m+c^m}{a^{m-1}+b^{m-1}+c^{m-1}}\geq \frac{a^n+b^n+c^n}{a^{n-1}+b^{n-1}+c^{n-1}} \implies$$

$$\left(\frac{a^m+b^m+c^m}{a^{m-1}+b^{m-1}+c^{m-1}}\right)^n\geq \left(\frac{a^n+b^n+c^n}{a^{n-1}+b^{n-1}+c^{n-1}}\right)^n$$

and also we have

$$\left(\frac{a^n+b^n+c^n}{a^{n-1}+b^{n-1}+c^{n-1}}\right)^n \geq \frac{a^n+b^n+c^n}{3} \iff \sqrt[n]{\frac{a^n+b^n+c^n}{3}} \geq$$
$$\geq \sqrt[n-1]{\frac{a^{n-1}+b^{n-1}+c^{n-1}}{3}}.$$

Hence

$$\left(\frac{a^m+b^m+c^m}{a^{m-1}+b^{m-1}+c^{m-1}}\right)^n \geq \frac{a^n+b^n+c^n}{3}.$$

Since $\left(\frac{a^n+b^n+c^n}{a^{n-1}+b^{n-1}+c^{n-1}}\right)^{n+1} \geq \frac{a^{n+1}+b^{n+1}+c^{n+1}}{3}, n \geq 3$ *and*

$$\left(\frac{a^m+b^m+c^m}{a^{m-1}+b^{m-1}+c^{m-1}}\right)^n \geq \frac{a^n+b^n+c^n}{3}, m \geq n$$

then, inequality $\left(\frac{a^m+b^m+c^m}{a^{m-1}+b^{m-1}+c^{m-1}}\right)^{n+1} \geq \frac{a^{n+1}+b^{n+1}+c^{n+1}}{3}$ *holds for any* $m \geq n \geq 3$.

Remark.
Note that Inequality **(1)** isn't right if $n=2$.Indeed, for $a=b=\frac{7}{24}, c=\frac{5}{12}$ we obtain

$$LHS = \left(\frac{2\left(\frac{7}{24}\right)^2+\left(\frac{5}{12}\right)^2}{2\cdot\frac{7}{24}+\frac{5}{12}}\right)^3 = \frac{1331}{32\,768}$$

$$RHS = \frac{2\left(\frac{7}{24}\right)^3+\left(\frac{5}{12}\right)^3}{3} = \frac{281}{6912}$$

and $\frac{1331}{32\,768} - \frac{281}{6912} = -\frac{31}{884\,736} < 0.$

Second solution. Power–means–inequality yields

$$a^m+b^m+c^m \geq (a^{m-1}+b^{m-1}+c^{m-1})^{\frac{m}{m-1}} 3^{-\frac{1}{m-1}}$$

that is

$$a^{m-1}+b^{m-1}+c^{m-1}\leq\left(a^m+b^m+c^m\right)^{\frac{m-1}{m}}3^{\frac{1}{m}}$$

whence

$$\left(\frac{a^m+b^m+c^m}{a^{m-1}+b^{m-1}+c^{m-1}}\right)^{n+1}\geq\left(\frac{a^m+b^m+c^m}{\left(a^m+b^m+c^m\right)^{\frac{m-1}{m}}3^{\frac{1}{m}}}\right)^{n+1}=$$
$$=\frac{\left(a^m+b^m+c^m\right)^{\frac{n+1}{m}}}{3^{\frac{n+1}{m}}}$$

Also Power–means yields

$$\frac{\left(a^m+b^m+c^m\right)^{\frac{n+1}{m}}}{3^{\frac{n+1}{m}}}\geq 3^{-\frac{n+1}{m}}\left[\left(a^{n+1}+b^{n+1}+c^{n+1}\right)^{\frac{m}{n+1}}3^{1-\frac{m}{n+1}}\right]^{\frac{n+1}{m}}=$$
$$=\frac{a^{n+1}+b^{n+1}+c^{n+1}}{3}$$

Paolo Perfetti

W11. (Solution by the proposer.) We have the following inequality:

$$\frac{x}{1-x^4}\geq\frac{5\sqrt[4]{5}}{4}x^2,\ (\forall)\,x\in(0,1)$$

or

$$\frac{1}{1-x^5}\geq\frac{5\sqrt[4]{5}}{4}x$$

If we denote $a=x\sqrt[4]{5}$ we have to prove that

$$\frac{1}{1-\frac{a^5}{5}}\geq\frac{5}{4}a \text{ or } a^5-5a+4\geq 0 \text{ or } \left(a-1\right)^2\left(a^3+2a^2+3a+4\right)\geq 0$$

From (1) we obtain

$$\int_0^t\frac{x}{1-x^4}dx\geq\frac{5\sqrt[4]{5}}{3}\cdot\frac{t^3}{3},\ \ (\forall)\,t\in(0,1)$$

or