József Wildt International Mathematical Competition

The Edition XXVth , 2015

The solution of the problems W.1 - W.34 must be mailed before 10. December 2015, to Mihály Bencze, str. Hărmanului 6, 505600 Săcele - Négyfalu, Jud. Braşov, Romania, E-mail: benczemihaly@yahoo.com

W1. Let $k\in N, k\geq 2$; $\alpha\geq 0$ and the sequence $(a_n)_{n\geq 1}$ defined by

$$a_b=\sqrt[k]{\alpha+\sqrt[k]{\alpha+...+\sqrt[k]{\alpha}}},\ (\forall)\,n\in N^*$$

a). If exist $p\in N^*$ such that $p^k\leq\alpha<(p+1)^k-(p+1)$, show that $[a_n]=p$, $(\forall)\,k\in N^*$
b). Show that
(i) exist $p\in N^*$ such that $p^k-k\leq\alpha<(p+1)^k-(p+1)$
(ii) if $\alpha\neq p^k-p$, exist $n_0\in N$ such that $[a_n]=p$, $\forall n\in N$, $n\geq n_0$
(iii) if $\alpha=p^k-p$, then $[a_n]=p-1$, $(\forall)\,n\in N^*$, where $[\cdot]$ denote the integer part.

Ovidiu T. Pop

W2. Let $a,b\in Q_+^*$ such that $a^2>b$ and $\sqrt{b}\notin Q$.
(i) Show that $\sqrt{a+\sqrt{b}}+\sqrt{a-\sqrt{b}}\in Q$ if and only if $(\exists)\,k\in Q_+$, $2a<k^2<4a$ and $b=ak^2-\frac{1}{4}k^4$
(ii) If $d\in Q_+$, $a^2>d$, $k\in Q_+$, $2a<k^2<4a$, $\sqrt{4a^2-k^2}\notin Q$ and $b=ak^2-\frac{1}{4}k^4$, then show that $\sqrt{a+\sqrt{b}}+\sqrt{a-\sqrt{d}}\in Q$ if and only if $d=b$.

Ovidiu T. Pop

W3. Let a,b,c be real positive numbers. Prove that

$$\sum_{\text{cyc}}\sqrt{\frac{4a^2+2b^2+4c^2+ab+bc}{3ac}}\geq 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}}\cdot$$
$$\cdot\sqrt{1+\frac{(a+b)(b+c)(c+a)-8abc}{2\left(a(a+b)(a+c)+b(b+c)(b+a)+c(c+a)(c+b)\right)}}$$

Paolo Perfetti

W4. Evaluate

$$\int_0^{+\infty}\frac{(y+1)\ln^2(1+y)}{(4y^2+8y+5)^{3/2}}dy$$

Paolo Perfetti

W5. If $f:R\to(1,+\infty)$, $g:R\to R$ are continuous functions and $y_n=\sqrt[n]{(2n-1)!!F_n}$, $n\in N^*\backslash\{1\}$ where F_n is the n^{th} Fibonacci number, then compute

$$\lim_{n\to\infty}\int\limits_{y_n}^{y_{n+1}}\frac{(f(x-y_n))^{g(y_{n+1}-x)}}{(f(y_{n+1}-x))^{g(x-y_n)}+(f(x-y_n))^{g(y_{n+1}-x)}}dx$$

D.M. Bătineţu-Giurgiu and Neculai Stanciu

W6. If $f:R\to(1,+\infty)$, $g:R\to R$ are continuous functions and $y_n=\sqrt[n]{n!L_n}$, $n\in N^*\backslash\{1\}$ where L_n is the n^{th} Lucas number, then compute

$$\lim_{n\to\infty}\int\limits_{y_n}^{y_{n+1}}\frac{(f(x-y_n))^{g(y_{n+1}-x)}}{(f(y_{n+1}-x))^{g(x-y_n)}+(f(x-y_n))^{g(y_{n+1}-x)}}dx$$

D.M. Bătineţu-Giurgiu and Neculai Stanciu

W7. Let $x \in R$ and $A(x) = \begin{pmatrix} x+1 & 1 & ... & 1 & 1 \\ 1 & x+1 & ... & 1 & 1 \\ ... & ... & ... & ... & ... \\ 1 & 1 & ... & x+1 & 1 \\ 1 & 1 & ... & 1 & x+1 \end{pmatrix} \in M_n(R)$. Compute

$$A(0) \cdot A(1) \cdot A(2) \cdot A(3).$$

D.M. Bătineţu-Giurgiu and Neculai Stanciu

W8. Let $sf(n) := 1!2!...n!$ (superfactorial). Prove that

$$\lim_{n\to\infty} \left(\frac{(n+1)\sqrt{n+1}}{\sqrt[(n+1)^2]{sf(n+1)}} - \frac{n\sqrt{n}}{\sqrt[n^2]{sf(n)}} \right) = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt[n^2]{sf(n)}} = e^{\frac{3}{4}}.$$

Arkady Alt

W9. Let $(x_n)_{n\geq 0}$ be sequence of complex numbers defined recursively

$$x_{n+1} = \frac{1}{k}(x_n + x_{n-1} + x_{n-2} + ... + x_{n-k+1}), n \geq k-1.$$

Determine $\lim_{n\to\infty} x_n$.

As a variant.

Let $(x_n)_{n\geq 0}$ be sequence of complex numbers defined recursively

$$x_{n+1} = \frac{1}{k}(x_n + x_{n-1} + x_{n-2} + ... + x_{n-k+1}), n \geq k-1.$$

Prove that

$$\lim_{n\to\infty} x_n = \frac{kx_{k-1} + (k-1)x_{k-2} + ... + 2x_1 + x_0}{\binom{k}{2}}.$$

Arkady Alt

W10. Let m, n be positive integer numbers such that $m \geq n \geq 3$. Prove that for any positive real a, b and c holds inequality

$$\left(\frac{a^m + b^m + c^m}{a^{m-1} + b^{m-1} + c^{m-1}} \right)^{n+1} \geq \frac{a^{n+1} + b^{n+1} + c^{n+1}}{3}.$$

Arkady Alt

W11. Let be $x, y, z > 0$ such that $\mathrm{x}^3 + y^3 + z^3 = 1$. Prove that

$$\left(1+x^2\right)\left(1+y^2\right)\left(1+z^2\right) \geq e^{\frac{5\sqrt[4]{5}}{2}} \left(1-x^2\right)\left(1-y^2\right)\left(1-z^2\right)$$

Marius Drăgan

W12. Let ABC be a triangle and M an interior point. Denote r_1, r_2, r_3 the radii of the incircle circles of BMC, CMB, AMB triangles. Prove that

$$\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} \geq 12 + 6\sqrt{3}$$

Generalization.

Marius Drăgan

W13. If $f, g : [0,1] \to (0, +\infty)$ are continuously functions and $(a_n)_{n\geq 1}$ is defined by

$$a_n = \int\limits_0^1 g(x) \sqrt[n]{f(x)} dx, a = \int\limits_0^1 g(x)\, dx$$

then prove that

$$\lim_{n\to\infty} \left(\frac{a_n}{a}\right)^n = e^{\frac{1}{a}\int\limits_0^1 g(x)\ln f(x) dx}$$

Nicolae Papacu

W14. If $a, b, c > 0$ such that $a \leq b \leq c$ then $\frac{3}{2^{n+1}} \leq \sum \frac{a^n+b^n}{(a+b)^n} \leq \frac{3}{2^n}\left(\sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}}\right)^n + 3\left(\frac{1}{2^{n-1}} - 1\right)$ for all $n \geq 1$.

Nicolae Papacu

W15. Calculate

$$\int_0^\infty \frac{(1-\sin ax)(1-\cos bx)}{x^2} \mathrm{d}x,$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R}^*$.

Ovidiu Furdui

W16. Let $p \geq 1$ be an integer. Calculate

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k} - \ln\left(k + \frac{1}{p}\right) - \gamma + \frac{2-p}{2pk}\right).$$

Ovidiu Furdui and Alina Sîntămărian

W17. Let $k > 0$ be a real number. Calculate

$$\int_0^1 \int_0^1 x^k \left\{\frac{1}{xy}\right\} \mathrm{d}x\mathrm{d}y,$$

where $\{a\}$ denotes the fractional part of a.

Ovidiu Furdui

W18. Let n be a positive integer. Prove that

$$1 + \left(\sum_{k=1}^{n} \frac{F_k \binom{n}{k}}{\sqrt{F_n F_{n+1}}}\right)^2 \leq \binom{2n}{n},$$

where F_n is the n^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$ and for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

José Luis Díaz-Barrero

W19. Let $z_1, z_2, \ldots, z_n$ be distinct nonzero complex numbers. For all $n \geq 3$, prove that

$$\sum_{k=1}^{n} \frac{1 + z_k^{n-1}}{z_k^2} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{z_k - z_j} = \sum_{k=1}^{n} \frac{1}{z_1 \ldots z_k^2 \ldots z_n}.$$

José Luis Díaz-Barrero

W20. Let F be the point of tangency of the nine-point circle and inscribed circle corresponding the triangle ABC. Show that

$$aFA^2 + bFB^2 + cFC^2 = 2sr\,(2R + r)$$

Nicuşor Minculete

W21. Let $A_1, A_2, ..., A_n$ be the vertices of the convex ploygon, $n \geq 3$, and M a point interior to the polygon. We note with R_k the distance from M to the vertices A_k and we note with r_k the distances from M to the sides $[A_k A_{k+1}]$, where $k = \overline{1, n}$ and $A_{n+1} = A_1$ and $R_{n+1} = R_1$. Prove that

$$\sum_{k=1}^{n} \frac{R_k + R_{k+1}}{r_k} \geq \frac{2n}{\cos \frac{n}{2}}$$

Nicuşor Minculete

W22. Given an infinite dimensional Banach space $(X, |\cdot|)$, construct a new norm which is not equivalent to $|\cdot|$, yet forms a Banach space with X.

József Kolumbán

W23. Let be $m, n \in N$. Solve the following system in N:

$$\begin{cases} x_1 + x_2 + \cdots x_m = m \cdot y_1 y_2 \cdots y_n \\ y_1 + y_2 + \cdots y_n = n \cdot x_1 x_2 \cdots x_m. \end{cases}$$

Kramer Alpár Vajk

W24. Prove the inequality

$$\frac{n!\,(n+1)!}{(2n)!} \leq \sum_{k=0}^{n} \frac{1}{\binom{n}{k}^2},\ n \in N^*$$

Study the convergence if the series

$$\sum_{k=1}^{n} \frac{1}{\binom{n}{k}^2}$$

Laurenţiu Modan

W25. Let I be a multiplicatively convex interval. If the continuous function $f : I \to (0, +\infty)$ verifies the following inequality

$$(f(x)\, f(y)\, f(z))^4 \left(f\left(\sqrt[3]{xyz}\right)\right)^6 \geq$$

$$\geq \left(f\left(\sqrt[3]{x^2 y}\right) f\left(\sqrt[3]{xy^2}\right) f\left(\sqrt[3]{y^2 z}\right) f\left(\sqrt[3]{yz^2}\right) f\left(\sqrt[3]{z^2 x}\right) f\left(\sqrt[3]{zx^2}\right)\right)^3$$

for all $x, y, z \in I$, then f is multiplicatively convex on I.

Vlad Ciobotariu Boer

W26. Let $f : I \subset R \to R$ be a twice differentiable function on I such that f'' is bounded on $[a, b]$, where $a, b \in I$, $a < b$. Prove that

$$\left| \frac{1}{b-a} \int_a^b f(u)\, du - \frac{1}{2}\left(\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right)\right) + \frac{b-a}{48}\left(f'(b) - f'(a)\right) \right| \leq$$

$$\leq \frac{\sqrt{3}}{216}\,(b-a)^2\, \|f''\|_{+\infty}$$

Vlad Ciobotariu Boer

W27. Find all positive functions f on R^+, which satisfy functional equation

$$f(x) = f(x(1-y)f^2(xy))f(xy)$$

for all $x \in R^+$ and $y \in (0,1)$.

Pál Péter Dályay

W28. Denote m_a, m_b, m_c the lengths of the medians in triangle ABC of sides $BC = a$, $CA = b$, $AB = c$. Prove that

$$\sum m_a m_b \leq \frac{1}{4}\left(5s^2 - 3r^2 - 12Rr\right)$$

Dorin Andrica

W29. Consider the sequence of polynomials $(P_n)_{n\geq 1}$ recursively defined by

$$P_{k+1}(x) = (x^a - 1)P_k'(x) - (k+1)P_k(x)$$

where $k = 1, 2, ...,$ $P_1(x) = x^{a-1}$ and $a \geq 2$ is a positive integer.
1). Finde the degree of P_k
2). Determine $P_k(0)$

Dorin Andrica

W30. If $a_k \in [0,1]\,(k = 1, 2, ..., n)$, then

$$\sum \frac{a_1^{n-1}}{(n-1)(a_2^n + a_3^n + ... + a_n^n) + 2n - 1} \leq \frac{1}{n}.$$

Mihály Bencze

W31. If $a_k > 0$ $(k = 1, 2, ..., n)$ then

$$(n-1)\sum_{cyclic} \frac{a_1}{a_2} + \sum_{cyclic} \frac{a_2^{n-2}}{a_3 a_4 ... a_n} \geq \frac{n\sum\limits_{k=1}^{n} a_k}{\sqrt[n]{\prod\limits_{k=1}^{n} a_k}}$$

Mihály Bencze

W32. If $x_k > 0$ $(k = 1, 2, ..., n)$ then

$$\sum_{cyclic} \sqrt{1 + \frac{6}{x_1 + x_2 + x_3} + \frac{3}{x_1x_2 + x_2x_3 + x_3x_1}} \geq n + \frac{n^2}{\sum\limits_{k=1}^{n} x_k}$$

Mihály Bencze

W33. Prove that the equation

$$x^4 + (yz)^4 = 2\left(t^8 + 6t^4 + 1\right)$$

have infinitely many solutions in Z. Solve the given equation in N.

Mihály Bencze

W34. Compute

$$\sum_{k=1}^{\infty}\left(\sum_{n=0}^{\infty}\frac{(k-1)!}{2^k\prod\limits_{j=1}^{k+1}(2n+j)}\right)$$

and

$$\sum_{k=1}^{\infty}\left(\sum_{n=0}^{\infty}\frac{(k-1)!}{2^k\prod\limits_{j=1}^{k+1}(2n+1+j)}\right).$$

Pál Péter Dályay