$$f(x) = g(x) + g'(x) + g''(x)$$

By Rolle's theorem there is a value $c \in (a, b)$ such that

$$1 = \frac{1}{b-a} \ln\left(\frac{f(b)}{f(a)}\right) = \frac{\ln(f(b)) - \ln(f(a))}{b-a} = \frac{d}{dx} \ln(f(x))\,|_{x=c} = \frac{f'(c)}{f(c)}$$

So $0 = f'(c) - f(c) = g'''(c) - g(c)$ which is equivalent to the claimed equality.

Albert Stadler

Fourth solution.

$$\left(\frac{f(b)}{f(a)}\right) = b - a \iff \frac{\ln f(b) - \ln f(a)}{b-a} = 1$$

and the Lagrange's theorem or the mean–value–theorem yields the existence of a point $c \in (a, b)$ such that

$$(\ln(f(x))'\big|_{x=c} = 1 \iff \frac{f'(c)}{f(c)} = 1 \iff f'(c) = f(c)$$

$$f'(c) = \frac{1 + \frac{c}{\sqrt{1+c^2}}}{c + \sqrt{1+c^2}} - \frac{c}{(1+c^2)^{\frac{3}{2}}} - \frac{1}{(1+c^2)^{\frac{3}{2}}} + \frac{3c^2}{(1+c^2)^{\frac{5}{2}}} =$$

$$= \frac{1}{\sqrt{1+c^2}} - \frac{c+1}{(1+c^2)^{\frac{3}{2}}} + \frac{3c^2}{(1+c^2)^{\frac{5}{2}}} =$$

The equation $f'(c) = f(c)$ yields

$$\ln(c + \sqrt{1+c^2}) = -\frac{1}{(1+c^2)^{\frac{3}{2}}} + \frac{3c^2}{(1+c^2)^{\frac{5}{2}}}$$

that is

$$2c^2 = 1 + (1+c^2)^{5/2} \ln(c + \sqrt{1+c^2})$$

Paolo Perfetti

W6. Solution by the proposer. Let $S(\mathbf{x}_{\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}}$ if series converges and $S_f(x_{\mathbb{N}}) = \infty$ if it diverges.
Let $\widetilde{D}_1 = \{x_{\mathbb{N}} \mid x_{\mathbb{N}} \in D_1 \text{ and } S(x_{\mathbb{N}}) \neq \infty\}$ .Since $\widetilde{D}_1$ isn't empty (because for for instance if $x_n = q^{n-1}, n \in \mathbb{N}$, where $q \in (0, 1)$, we have

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} =$$

$$= \sum_{n=1}^{\infty} \frac{q^{3(n-1)}}{q^{n-1} + 4q^n} = \sum_{n=1}^{\infty} \frac{q^{2(n-1)}}{1+4q} = \frac{1}{(1+4q)(1-q^2)}$$

then $\inf \{S(\mathbf{x}_{\mathbb{N}}) \mid \mathbf{x}_{\mathbb{N}} \in D_1\} = \inf \left\{S(\mathbf{x}_{\mathbb{N}}) \mid \mathbf{x}_{\mathbb{N}} \in \widetilde{D}_1\right\}$.

Let $S := \inf \left\{S(\mathbf{x}_{\mathbb{N}}) \mid \mathbf{x}_{\mathbb{N}} \in \widetilde{D}_1\right\}$. For any $\mathbf{x}_{\mathbb{N}} \in \widetilde{D}_1$ we have

$$S(\mathbf{x}_{\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} =$$

$$= \frac{1}{1+4x_2} + \sum_{n=2}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} = \frac{1}{1+4x_2} + x_2^2 \sum_{n=1}^{\infty} \frac{y_n^3}{y_n + 4y_{n+1}} = \frac{1}{1+4x_2} + x_2^2 S(\mathbf{y}_{\mathbb{N}}),$$

where $y_n := \dfrac{x_{n+1}}{x_2}, n \in \mathbb{N}$.

Since $\mathbf{y}_{\mathbb{N}} \in \widetilde{D}_1$ ($1 = y_1 > y_2 > .... > y_n > ...$ and $S(\mathbf{y}_{\mathbb{N}}) = \dfrac{S(\mathbf{x}_{\mathbb{N}})}{x_2^2} - \dfrac{1}{1+4x_2}$) then $S(\mathbf{y}_{\mathbb{N}}) \geq S$ and, therefore, $S(\mathbf{x}_{\mathbb{N}}) \geq \dfrac{1}{1+4x_2} + x_2^2 S \implies S \geq \dfrac{1}{1+4x_2} + x_2^2 S \iff$ $S \geq \dfrac{1}{(1+4x_2)(1-x_2^2)}$.

We will find $\mu := \max\limits_{x \in (0,1)} h(x)$, where

$$h(x) := (1+4x)(1-x^2) = -4x^3 - x^2 + 4x + 1$$

Since $h'(x) = -12x^2 - 2x + 4 = -2(3x+2)(2x-1)$ then

$\mu = \max\limits_{x \in (0,1)} h(x) = h\left(\dfrac{1}{2}\right) = \dfrac{9}{4}$ and, therefore, $S(\mathbf{x}_{\mathbb{N}}) \geq \dfrac{1}{\mu} = \dfrac{4}{9}$.

Since $S(\mathbf{x}_{\mathbb{N}}) = \dfrac{1}{(1+4q)(1-q^2)}$ for $x_n = q^{n-1}, n \in \mathbb{N}, q \in (0,1)$, then for $q = \dfrac{1}{2}$ we obtain

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} = \frac{1}{\left(1 + 4 \cdot \frac{1}{2}\right)\left(1 - \left(\frac{1}{2}\right)^2\right)} = \frac{4}{9}.$$

Second solution. If $x_k = 2^{-k+1}$ the equality occurs. Indeed

$$\sum_{n=1}^{\infty} \frac{2^{-3n+3}}{2^{-n+1} + 4 \cdot 2^{-n}} = \frac{4}{3} \sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{4}{3} \frac{1}{4} \frac{1}{1 - \frac{1}{4}} = \frac{4}{9}$$

Now consider the sequence $y_1 = 1$, $y_2 = \frac{1}{2} + \delta$, $0 < \delta < 1/2$, $y_k = 2^{-k+1}$ for $k \geq 3$.

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} - \sum_{n=1}^{\infty} \frac{y_n^3}{y_n + 4y_{n+1}} = \frac{4}{9} - \sum_{n=1}^{\infty} \frac{y_n^3}{y_n + 4y_{n+1}} =$$

$$= \frac{x_1^3}{x_1 + 4x_2} + \frac{x_2^3}{x_2 + 4x_3} - \frac{y_1^3}{y_1 + 4y_2} - \frac{y_2^3}{y_2 + 4y_3} =$$

$$= \frac{1}{1 + 4\frac{1}{2}} + \frac{\frac{1}{8}}{\frac{1}{2} + 4\frac{1}{4}} - \frac{1}{1 + 4(\frac{1}{2} + \delta)} - \frac{(\frac{1}{2} + \delta)^3}{\frac{1}{2} + \delta + 4\frac{1}{4}} =$$

$$= \frac{5}{12} - \frac{1}{3}\frac{1}{1 + \frac{4}{3}\delta} - \left(\frac{1}{2} + \delta\right)^3 \frac{2}{3}\frac{1}{1 + \frac{2}{3}\delta} =$$

$$= \frac{5}{12} - \frac{1}{3}\left(1 - \frac{4}{3}\delta\right) - \frac{1}{8}\frac{2}{3}\left(1 - \frac{2}{3}\delta\right) + O(\delta^2) = \frac{1}{2}\delta + O(\delta^2) > 0$$

Contradicting the statement.

Paolo Perfetti

Third solution. Q$x_1 = 1$, and sequence $\{x_n\}$ is the increasing geometric progression,

$\therefore x_n = 1 \times q^{n-1}$ $(0 < q < 1)$

Then $y_n = \frac{x_n^3}{x_n + 4x_{n+1}} = \frac{q^{3n-3}}{q^{n-1} + 4q^n} = \frac{q^{2n-2}}{1+4q}$

So

$$\sum_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} = \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{q^{2n-2}}{1 + 4q} = \sum_{n=1}^{\infty} \frac{1}{1 + 4q} \times \frac{1 - q^{2n}}{1 - q^2}$$

Q$0 < q < 1$, $n \to \infty$, $\therefore 1 - q^{2n} \to 1$,

$\therefore \sum\limits_{n=1}^{\infty} \frac{x^3}{x_n + 4x_{n+1}} = \sum\limits_{n=1}^{\infty} \frac{1}{1+4q} \times \frac{1}{1-q^2}$

$f(q) = (1 + 4q)(1 - q^2) = 1 - q^2 + 4q - 4q^3$ $(0 < q < 1)$

$f'(q) = -2q + 4 - 12q^2$

Let $f'(q) = -2q + 4 - 12q^2 = 0$, $\therefore q_1 = -\frac{2}{3}$ (round) $q_2 = \frac{1}{2}$

$\therefore$ In interval (0,1), maximum values for q is $f\left(\frac{1}{2}\right) = \frac{9}{4}$. If and only if $q = \frac{1}{2}$,

$\therefore \frac{9}{4} \geq \frac{1}{1+4q} \times \frac{1}{1-q^2}$, $\therefore (1 + 4q) \times (1 - q^2) \leq \frac{9}{4}$, $\therefore \sum\limits_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} \leq \frac{9}{4}$.

When $q = \frac{1}{2}$ $\sum\limits_{n=1}^{\infty} \frac{x_n^3}{x_n + 4x_{n+1}} = \frac{9}{4}$. At this time $x_n = \frac{1}{2^{n-1}}$.

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Fourth solution. By the Cauchy-Schwarz-inequality

$$\sum_{n=1}^{\infty}\frac{x_n^3}{x_n+4x_{n+1}}\cdot\sum_{n=1}^{\infty}x_n\left(x_n+4x_{n+1}\right)=$$

$$=\sum_{n=1}^{\infty}x_n^2\frac{x_n}{x_n+4x_{n+1}}\cdot\sum_{n=1}^{\infty}x_n^2\frac{x_n+4x_{n+1}}{x_n}\geq\left(\sum_{n=1}^{\infty}x_n^2\right)^2$$

Again, by the Cauchy-Schwarz-inequality

$$\sum_{n=1}^{\infty}\frac{x_n^3}{x_n+4x_{n+1}}\geq\frac{\left(\sum\limits_{n=1}^{\infty}x_n^2\right)^2}{\sum\limits_{n=1}^{\infty}x_n\left(x_n+4x_{n+1}\right)}=\frac{\left(\sum\limits_{n=1}^{\infty}x_n^2\right)^2}{\sum\limits_{n=1}^{\infty}x_n^2+4\sum\limits_{n=1}^{\infty}x_nx_{n+1}}\geq$$

$$\geq\frac{\left(\sum\limits_{n=1}^{\infty}x_n^2\right)^2}{\sum\limits_{n=1}^{\infty}x_n^2+4\sqrt{\sum\limits_{n=1}^{\infty}x_n^2\cdot\sum\limits_{n=1}^{\infty}x_{n+1}^2}}=\frac{s^2}{s+4\sqrt{s\left(s-1\right)}}$$

where $s=\sum\limits_{n=1}^{\infty}x_n^2$.

The inequality $\frac{s^2}{s+4\sqrt{s(s-1)}}\geq\frac{4}{9}$ is equivalent to each of

$$\left(s^2-\frac{4}{9}s\right)^2\geq\frac{256}{81}s\left(s-1\right)\text{ and }\frac{s\left(3s-4\right)^2\left(16+9s\right)}{81}\geq0,$$

which obviously holds true. So $\sum\limits_{n=1}^{\infty}\frac{x_n^3}{x_n+4x_{n+1}}\geq\frac{4}{9}$.

Equality holds true only if $\frac{x_n+4x_{n+1}}{x_n}$ is constant and $s=\frac{4}{3}$, which means that $\frac{x_{n+1}}{x_n}=c$ for some constant c implying $x_n=c^{n-1}$ and $s=\sum\limits_{n=1}^{\infty}c^{n-2}=\frac{1}{1-c^2}=\frac{4}{3}$. So $c=\frac{1}{2}$ and $x_n=\frac{1}{2^{n-1}}$, $n=1,2,\ldots$

Indeed,

$$\sum_{n=1}^{\infty}\frac{x_n^3}{x_n+4x_{n+1}}=\sum_{n=0}^{\infty}\frac{\frac{1}{8^n}}{\frac{1}{2^n}+\frac{2}{2^n}}=\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{4^n}=\frac{4}{9}.$$

Albert Stadler

W7. Solution by the proposer. Let $F := [ABC]$. Since $[A_1CB_1I] = [A_1AB] - [IAB_1]$ and $\frac{CA_1}{BC} = \frac{b}{b+c}, \frac{AB_1}{AC} = \frac{c}{a+c}, \frac{IB_1}{BB_1} = \frac{b}{a+b+c}$ then

$$[A_1AC] = \frac{bF}{b+c}, [ABB_1] = \frac{cF}{a+c}, [IAB_1] = \frac{b\,[ABB_1]}{a+b+c} = \frac{bcF}{(a+b+c)(a+c)},$$

$$[A_1CB_1I] = \frac{bF}{b+c} - \frac{bcF}{(a+b+c)(a+c)} = \frac{bF\left((a+c)^2 + ab + bc - bc - c^2\right)}{(a+b+c)(a+c)(b+c)} =$$

$$= \frac{Fab\,(2c+a+b)}{(a+b+c)(a+c)(b+c)} = \frac{F\left((a+b)^2 - c^2\right)(2c+a+b)}{2\,(a+b+c)(a+c)(b+c)} =$$

$$= \frac{F\,(a+b-c)(2c+a+b)}{2\,(a+c)(b+c)}.$$

Thus

$$\frac{[A_1CB_1I]}{F} = \frac{(a+b-c)(2c+a+b)}{2\,(a+c)(b+c)}.$$

Now we will find $\max \frac{(a+b-c)(2c+a+b)}{2\,(a+c)(b+c)}$.

Due to homogeneity of $\frac{(a+b-c)(2c+a+b)}{(a+c)(b+c)}$ we can assume that $c = 1$.

Since $a^2 + b^2 = 1$ then, denoting $t := a + b$ obtain that $t \le \sqrt{2}$ ($\iff a + b \le \sqrt{2\,(a^2+b^2)}$),

$$(a+1)(b+1) = 1 + t + \frac{t^2-1}{2} = \frac{(t+1)^2}{2}, \frac{(a+b-c)(2c+a+b)}{2\,(a+c)(b+c)} =$$

$$= \frac{(t-1)(2+t)}{2\,(a+1)(b+1)} = \frac{(t-1)(2+t)}{(t+1)^2}$$

and, therefore,

$$\max\left(\frac{(a+b-c)(2c+a+b)}{2\,(a+c)(b+c)}\right) = \max_{0<t\le\sqrt{2}}\left(\frac{(t-1)(t+2)}{(t+1)^2}\right).$$

Since $t + 1 \le \sqrt{2} + 1 \iff \frac{1}{t+1} \ge \sqrt{2} - 1$ then

$$\frac{(t-1)(t+2)}{(t+1)^2} = \frac{t^2+t-2}{(t+1)^2} = 1 - \frac{1}{t+1} -$$

$$-\frac{2}{(t+1)^2} \le 1 - \left(\sqrt{2}-1\right) - 2\left(\sqrt{2}-1\right)^2 = 2 - \sqrt{2} - 6 + 4\sqrt{2} = 3\sqrt{2} - 4$$

and equaliy occurs iff $t = \sqrt{2}$. Thus, $\max \frac{[A_1CB_1I]}{F} = 3\sqrt{2} - 4$ and can be attained only iff $a = b = \frac{\sqrt{2}c}{2}$ because

$$\begin{cases} a^2 + b^2 = c^2 \\ a + b = \sqrt{2}c \end{cases} \Longleftrightarrow \begin{cases} a^2 + b^2 = c^2 \\ a = b \end{cases}.$$

Second solution. $2\alpha + 2\beta = \frac{\pi}{2}$, $\alpha + \beta = \frac{\pi}{4}$, $a^2 + b^2 = c^2$
In the $\triangle ABI$, set $BI = x$, $AI = y$, set $\measuredangle BAI = \measuredangle\alpha$, $\measuredangle ABI = \measuredangle\beta$,

$$\frac{x}{\sin\alpha} = \frac{y}{\sin\beta} = \frac{c}{\sin(\pi - \alpha - \beta)} = \frac{c}{\sin(\alpha + \beta)}$$

$\therefore x = \sqrt{2}\sin\alpha \times c$, $y = \sqrt{2}\sin\beta\times$
$S_{\triangle ABI} = \frac{1}{2} \times 2\sin\alpha\sin\beta c^2 \times \sin(\pi - \alpha - \beta) - \frac{2}{\sqrt{2}}\sin\alpha\sin\beta c^2\times$
Similarly avaible $S_{\triangle ABA_1} = \frac{1}{2}c^2 \times \frac{\sin\alpha\sin 2\beta}{\sin(\alpha+2\beta)}$, $S_{\triangle ABB_1} = \frac{1}{2}c^2 \times \frac{\sin 2\alpha\sin\beta}{\sin(2\alpha+\beta)}$
Let S be A_1CB_1I except the quadrilateral area,

$$\therefore S = \frac{1}{2}c^2 \times \left[\frac{2\sin\alpha\sin\beta}{\sin(2\alpha + \beta)} + \frac{\sin\alpha\sin 2\beta}{\sin(\alpha + 2\beta)} - \sqrt{2}\sin\alpha\sin\beta\right],$$

$S_{\triangle ABC} = \frac{1}{2}c^2 \times \sin 2\alpha\sin 2\beta$.
Now to make the minimum $\frac{S}{S_{\triangle ABC}}$, so the proportion of quadrilateral is maximized

$$\frac{S}{S_{\triangle ABC}} = \frac{\sin\beta}{\sin(2\alpha + \beta)\sin 2\beta} + \frac{\sin\alpha}{\sin(\alpha + 2\beta)\sin 2\alpha} - \frac{\sqrt{2}\sin\alpha\sin\beta}{\sin 2\alpha\sin 2\beta} =$$

$$= \frac{1}{2\sin(2\alpha + \beta)\cos\beta} + \frac{1}{2\sin(\alpha + 2\beta)\cos\alpha} - \frac{\sqrt{2}}{4\cos\alpha\cos\beta} =$$

$$= \frac{1}{2\cos\alpha\cos\beta} + \frac{1}{2\cos\beta\cos\alpha} - \frac{\frac{\sqrt{2}}{2}}{2\cos\alpha\cos\beta} = \frac{2 - \frac{\sqrt{2}}{2}}{2\cos\alpha\cos\beta}$$

Now to make the $f(\alpha) = 2\cos\alpha\cos\beta$ max, $\alpha + \beta = \frac{\pi}{4}$,

$$\therefore f(\alpha) = 2\cos\alpha\cos\left(\frac{\pi}{4} - \alpha\right) = 2\cos\left(\frac{\sqrt{2}}{2}\cos\alpha + \frac{\sqrt{2}}{2}\sin\alpha\right) =$$

$$= \sqrt{2}\cos^2\alpha + \sqrt{2}\sin\alpha\cos\alpha = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(2\cos^2\alpha - 1\right) + \frac{\sqrt{2}}{2}\sin 2\alpha =$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cos 2\alpha + \frac{\sqrt{2}}{2}\sin 2\alpha = \frac{\sqrt{2}}{2} + \sin\left(2\alpha + \frac{\pi}{4}\right)$$

$\max f(\alpha) = \frac{\sqrt{2}}{2} + 1$, at this time $\alpha = \frac{\pi}{8}$ $\min \frac{S}{S_{\triangle ABC}} = \frac{2-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}+1} = 5 - 3\sqrt{2}$.

So $\frac{S_{[A_1CB_1I]}}{S_{\triangle ABC}} = 1 - \frac{S}{S_{\triangle ABC}} = 1 - \left(5 - 3\sqrt{2}\right) = 3\sqrt{2} - 4.$

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Third solution. We can assume that $AC = 1$. We put $\alpha = \angle BAC = (=$ angle at $A)$. Then $area\left(\triangle ABC\right) = \frac{1}{2}\tan\left(\alpha\right)$, $area\left(\triangle AA_1C\right) = \frac{1}{2}\tan\left|\frac{\alpha}{2}\right|$, $area\left(\triangle BCB_1\right) = \frac{1}{2}\tan^2\left(\alpha\right)\tan\left|\frac{\frac{\pi}{2}-\alpha}{2}\right|$,

$$area\left(\triangle ABI\right) = \frac{1}{2}AI \cdot BI\sin\left|\pi - \frac{\alpha}{2} - \frac{\frac{\pi}{2}-\alpha}{2}\right| = \frac{1}{2}AI \cdot BI\frac{\sqrt{2}}{2} =$$

$$= \frac{1}{2}\frac{AB\sin\left|\frac{\frac{\pi}{2}-\alpha}{2}\right|}{\sin\left|\frac{3\pi}{4}\right|} \cdot \frac{AB\sin\left|\frac{\alpha}{2}\right|}{\sin\left|\frac{3\pi}{4}\right|}\sin\left|\frac{3\pi}{4}\right| =$$

$$= \frac{\sqrt{2}}{2}AB^2\sin\left|\frac{\frac{\pi}{2}-\alpha}{2}\right|\sin\left|\frac{\alpha}{2}\right| = \frac{\sqrt{2}}{2}\frac{\sin\left|\frac{\frac{\pi}{2}-\alpha}{2}\right|\sin\left|\frac{\alpha}{2}\right|}{\cos^2\alpha} = \frac{\left|\cos\left|\frac{\alpha}{2}\right| - \sin\left|\frac{\alpha}{2}\right|\right|\sin\left|\frac{\alpha}{2}\right|}{2\cos^2\left(\alpha\right)},$$

$area\left(\diamond A_1CB_1I\right) = area\left(\triangle AA_1C\right) + area\left(\triangle BCB_1\right) + area\left(\triangle ABI\right) - area\left(\triangle ABC\right)$. Then

$$\frac{area\left(\diamond A_1CB_1I\right)}{area\left(\triangle ABC\right)} =$$

$$= \frac{\frac{1}{2}\tan\left|\frac{\alpha}{2}\right| + \frac{1}{2}\tan^2\left(\alpha\right)\tan\left|\frac{\frac{\pi}{2}-\alpha}{2}\right| + \frac{\left|\cos\left|\frac{\alpha}{2}\right|-\sin\left|\frac{\alpha}{2}\right|\right|\sin\left|\frac{\alpha}{2}\right|}{2\cos^2(\alpha)} - \frac{1}{2}\tan\left(\alpha\right)}{\frac{1}{2}\tan\left(\alpha\right)} =$$

$$= \frac{\tan\left|\frac{\alpha}{2}\right|}{\tan\left(\alpha\right)} + \tan\left(\alpha\right)\frac{1 - \tan\left|\frac{\alpha}{2}\right|}{1 + \tan\left|\frac{\alpha}{2}\right|} + \frac{\left|\cos\left|\frac{\alpha}{2}\right| - \sin\left|\frac{\alpha}{2}\right|\right|\sin\left|\frac{\alpha}{2}\right|}{\sin\left(\alpha\right)\cos\left(\alpha\right)} - 1 =$$

$$= \frac{1 - \tan^2\left|\frac{\alpha}{2}\right|}{2} + \frac{2\tan\left|\frac{\alpha}{2}\right|}{2\cos\left(\alpha\right)} - 1 = \frac{1 - \tan^2\left|\frac{\alpha}{2}\right|}{2} + \frac{2\tan\left|\frac{\alpha}{2}\right|}{\left|1 + \tan\left|\frac{\alpha}{2}\right|\right|^2} + \frac{1 - \tan\left|\frac{\alpha}{2}\right|}{1 - \tan^2\left|\frac{\alpha}{2}\right|} - 1 =$$

$$\overset{u=\tan\left|\frac{\alpha}{2}\right|}{=} \frac{1 - u^2}{2} + \frac{2u}{\left(1 + u\right)^2} + \frac{1 + u^2}{2\left(1 + u\right)} - 1 = f\left(u\right)$$

We have

$$\frac{d}{du}\left|\frac{1 - u^2}{2} + \frac{2u}{\left(1 + u\right)^2} + \frac{1 + u^2}{2\left(1 + u\right)} - 1\right| = \frac{\left(1 - 2u - u^2\right)\left(3 + u + 2u^2\right)}{2\left(1 + u\right)^3}$$

So $f\left(u\right)$ gets minimal at the positive zero of $1 - 2u - u^2 = 0$ or at $u = \sqrt{2} - 1$. If $\tan\left(\frac{\alpha}{2}\right) = \sqrt{2} - 1$ then $\tan\left(\alpha\right) = 1$.

So $\alpha = \frac{\pi}{4}$ and thus the right triangle with the greatest ratio is the isosceles right triangle.

Albert Stadler

Fourth solution. We use the usual notations. If r is the inradius, then $r = \frac{a+b-c}{2}$.
By bisector theorem we obtain that $A_1C = \frac{ab}{b+c}$ and $B_1C = \frac{ab}{a+c}$.
Yields that

$$2\left[A_1CB_1I\right] = 2\left[A_1CI\right] + 2\left[CB_1I\right] = r\left(\frac{ab}{b+c} + \frac{ab}{a+c}\right)$$

so

$$\frac{\left[A_1CB_1I\right]}{[ABC]} = \frac{a+b-c}{2} \cdot \frac{a+b+2c}{(a+c)(b+c)}$$

Because we suspect that the maximum value of this ratio is reached within an isosceles right triangle, we demonstrate that

$$\frac{(a+b-c)(a+b+2c)}{(a+c)} \leq \frac{2\sqrt{2}}{3+2\sqrt{2}}$$

which yields that the greatest value is $\frac{\sqrt{2}}{3+2\sqrt{2}}$.
We have:

$$\frac{(a+b-c)(a+b+2c)}{(a+c)(b+c)} \leq \frac{2\sqrt{2}}{3+2\sqrt{2}} \Leftrightarrow$$

$$\Leftrightarrow \left(3+2\sqrt{2}\right)c(a+b) - 2\sqrt{2}c(a+b) \leq 2\sqrt{2}\left(a^2+b^2+ab\right) + \left(3+2\sqrt{2}\right)\left(a-2ab+b^2\right) \Leftrightarrow$$

$$\Leftrightarrow 3c(a+b) \leq 2\sqrt{2}\left(2a^2-ab+2b^2\right) + 3\left(a^2-2ab+b^2\right)$$

and squared we obtain

$$9(a+b)^2\left(a^2+b^2\right) \leq 8\left(2a^2-ab+2b^2\right)^2 + 9\left(a^2-2ab+b^2\right)^2 +$$

$$+12\sqrt{2}\left(2a^2-ab+2b^2\right)\left(a^2-2ab+b^2\right) \Leftrightarrow$$

$$\Leftrightarrow 32a^4 - 86a^3b + 108a^2b^2 - 86ab^3 + 32b^4 + 24\sqrt{2}a^4 - 60\sqrt{2}a^3b + 72\sqrt{2}a^2b^2 -$$

$$-60\sqrt{2}ab^3 + 24\sqrt{2}b^4 \geq 0 \Leftrightarrow (a-b)^2\left(16a^2-11ab+16b^2\right) + 6\sqrt{2}(a-b)^2\left(2a^2-ab+2b^2\right) \geq 0,$$

true, since $16a^2 - 11ab + 16b^2 0$, $2a^2 - ab + 2b^2 > 0$. We have equality iff $a = b$, i.e the triangle is isosceles right triangle.

Neculai Stanciu

W8. Solution by the proposer. Note that $\Delta\left(x^2, y^2, z^2\right) = (x+y+z)(x+y-z)(x-y+z)(-x+y+z)$ and for positive x, y, z we have equivalency

$$\Delta\left(x^2, y^2, z^2\right) > 0 \iff \begin{cases} x+y>z \\ y+z>x \\ z+x>y \end{cases}.$$

Due symmetry and homogeneity of $\Delta\left(a^n, b^n, c^n\right) > 0$ WLOG we assume that $a \geq b \geq 1$.
Then for any $n \in \mathbb{N}$ we have

$$\begin{cases} \Delta\left(a^{2n}, b^{2n}, c^{2n}\right) > 0 \\ a \geq b \geq c = 1 \end{cases} \iff \begin{cases} b^n + 1 > a^n \\ a \geq b \geq c = 1 \end{cases}.$$

Suppose that $a > b$, then

$$a^n = (b + (a-b))^n > b^n + n(a-b)b^{n-1} > b^n + n(a-b) > b^n + 1$$

for any $n > \dfrac{1}{a-b}$. It is contradict to $b^n + 1 > a^n$ which holds for any $n \in \mathbb{N}$.
Thus $a = b$ and, therefore, triangle should be isosceles with two equal sides, which not less then third one.
Let now $a = b \geq c$ then

$$\Delta\left(a^n, b^n, c^n\right) = 2a^n b^n + 2b^n c^n + 2c^n a^n - a^{2n} - b^{2n} - c^{2n} =$$

$$= 4c^n a^n - c^{2n} \geq 3c^{2n} > 0.$$

Second solution. 1).When $a = b = c$, $V\left(a^n, b^n, c^n\right) = V\left(a^n, a^n, a^n\right) = 2a^n a^n + 2a^n a^n + 2a^n a^n - (a^n)^2 - (a^n)^2 - (a^n)^2 = 6a^{2n} - 3a^{2n} = 3a^{2n} > 0$.

2). When $b = c > a$,
$V\left(a^n, b^n, c^n\right) = V\left(a^n, b^n, b^n\right) = 2a^n b^n + 2b^n b^n + 2d^n b^n - (a^n)^2 - (b^n)^2 - (b^n)^2 = 2b^{2n} + 4a^n b^n - a^{2n} - 2b^{2n} = 4a^n b^n - a^{2n} = a^n\left(4b^n - a^n\right)$, $Qb > a$, $\therefore 4b^n > a^n$, $\therefore a^n\left(4b^n - a^n\right) > 0$, $\therefore \left(a^n, b^n, c^n\right) > 0$.

Chen Jianan, Lee Xueliang, Guo Meiqun, Feng Xiaoling, kong Huimin and Zheng Nanmin

Third solution. By symmetry we can assume that $a \le b \le c$. We distinguish two cases:
a). $b < c$

$$\triangle(a^n, b^n, c^n) = -(a^n - b^n)^2 + 2c^n(a^n + b^n) - c^{2n} \le 2c^n(a^n + b^n) - c^{2n} =$$

$$= c^{2n}\left(-1 + 2\frac{a^n + b^n}{c^n}\right) < 0$$

if n is sufficiently big, since $\frac{a}{c-1} < 1$, $\frac{b}{c} < 1$.
b). $b = c$

$$\triangle(a^n, b^n, b^n) = 4a^n b^n - a^{2n} = a^n(4b^n - a^n) > 0,$$

for all natural numbers n. So $\triangle(a^n, b^n, c^n) > 0$ for any natural number n if and only if a, b, c form the sides of an isosceles triangle whose third side is smaller than the two legs or a, b, c form the sides of an equilateral triangle.

Albert Stadle

W9. Solution by the proposer. a) Follows immediately from inequality $s^2 \le 4R^2 + 4Rr + 3r^2$. Really,

$$R^2 - 4r^2 \ge \frac{1}{5}\cdot(s^2 - 27r^2) \iff s^2 \le 5R^2 + 7r^2 \iff$$

$$0 \le (4R^2 + 4Rr + 3r^2 - s^2) + (R - 2r)^2.$$

b) Recall ([1]), that a triple (R, r, s) of positive real numbers can determine a triangle, where R, r, and s be a circumradius, inradius and semiperimeter respectively iff $(R, r, s) \in \bar{\Delta} := \{(R, r, s) \mid R \ge 2r \text{ and } L(R, r) \le s^2 \le M(R, r)\}$ where

$$L(R, r) = 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R(R - 2r)}$$

and

$$M(R, r) = 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R(R - 2r)}.$$

Since a triangle is equilateral iff $R = 2r$ then set

$$\Delta := \left\{(R, r, s) \mid (R, r, s) \in \bar{\Delta} \text{ and } R \ne 2r\right\}$$

determine all non-equilateral triangles.
Thus,

$$\max K = \min_{(R,r,s)\in\Delta} \frac{R^2 - 4r^2}{s^2 - 27r^2} = \min_{R>2r}\left(\min_{s} \frac{R^2 - 4r^2}{s^2 - 27r^2}\right) = \min_{R>2r} \frac{R^2 - 4r^2}{M(R, r) - 27r^2} =$$

$$= \min_{R>2r} \frac{R^2 - 4r^2}{2R^2 + 10Rr - 28r^2 + 2(R-2r)\sqrt{R(R-2r)}} = \min_{R>2r} \frac{R+2r}{2R - 14r + 2\sqrt{R(R-2r)}} =$$

$$= \min_{R>2r} \frac{1 + \frac{2r}{R}}{2 - 7 \cdot \frac{2r}{R} + 2\sqrt{1 - \frac{2r}{R}}}.$$

Denoting $t := \sqrt{1 - \frac{2r}{R}}$ we obtain that $t \in (0,1)$ and, therefore,

$$K_* := \max K = \min_{t\in(0,1)} \frac{2-t^2}{9+2t-7t^2} = \frac{23+\sqrt{17}}{8},$$

because $\frac{23+\sqrt{17}}{128}$ is smallest real k for which equation $\frac{2-t^2}{9+2t-7t^2} = k$ have solution in $(0,1)$.

Indeed, if equation $\frac{2-t^2}{9+2t-7t^2} = k$ have solution then

$$2 - t^2 = k\left(9 + 2t - 7t^2\right) \iff (7k-1)\,t^2 - 2kt - 9k + 2 = 0$$

yields

$$k^2 + (7k-1)(9k-2) = 64k^2 - 23k + 2 \geq 0 \implies k \geq \frac{23+\sqrt{17}}{128}.$$

Since for $k_* := \frac{23+\sqrt{17}}{128}$ equation

$$\frac{2-t^2}{9+2t-7t^2} = k_*$$

have only solution

$$t_* = \frac{k_*}{7k_*-1} = \frac{23+\sqrt{17}}{33+7\sqrt{17}} = \frac{5-\sqrt{17}}{2} \in (0,1)$$

then

$$K_* = \min_{t\in(0,1)} \frac{2-t^2}{9+2t-7t^2} = \frac{2-t_*^2}{9+2t_*-7t_*^2} = \frac{23+\sqrt{17}}{128}.$$

So, for any triangle holds inequality

$$R^2 - 4r^2 \geq \frac{23+\sqrt{17}}{128}\left(s^2 - 27r^2\right)$$

and not exist constant $K > \frac{23+\sqrt{17}}{128}$ which provide inequality in (b).

c) $\lim\limits_{R\to 2r}\dfrac{R^2-4r^2}{s^2-27r^2}=\dfrac{2}{9}$ because

$$\frac{R^2-4r^2}{2R^2+10Rr-28r^2+2(R-2r)\sqrt{R(R-2r)}}\le$$

$$\le\frac{R^2-4r^2}{s^2-27r^2}\le\frac{R^2-4r^2}{2R^2+10Rr-28r^2-2(R-2r)\sqrt{R(R-2r)}}\iff$$

$$\frac{R+2r}{2R+14r+2\sqrt{R(R-2r)}}\le\frac{R^2-4r^2}{s^2-27r^2}\le\frac{R+2r}{2R+14r-2\sqrt{R(R-2r)}}\text{ and}$$

$$\lim_{R\to 2r}\frac{R+2r}{2R+14r+2\sqrt{R(R-2r)}}=\lim_{R\to 2r}\frac{R+2r}{2R+14r-2\sqrt{R(R-2r)}}=\frac{2}{9}.$$

[1] D.S.Mitrinovic,J.E.Pecaric,V.Volnec.Recent Advances In Geometric Inequalities.

Second solution. Let a,b and c be the sides of the triangle, and let $\triangle$ be its area. Then

$$s=\frac{a+b+c}{2},\triangle=\sqrt{s(s-a)(s-b)s-c},R=\frac{abc}{4\triangle},r=\frac{\triangle}{s}$$

Let $x=b+c-a, y=c+a-b, z=a+b-c$. Then $x\ge 0, y\ge 0, z\ge 0$, $x+y=2c$, $y+z=2a$, $z+x=2b$, $x+y+z=a+b+c$. We express s,R,r in terms of x,y,z: $s=\frac{x+y+z}{2}$, $\triangle=\frac{1}{4}\sqrt{xyz(x+y+z)}$, $R=\frac{abc}{4\triangle}=\frac{(x+y)(y+z)(z+x)}{8\sqrt{xyz(x+y+z)}}$, $r=\frac{\triangle}{s}=\frac{1}{2}\sqrt{\frac{xyz}{x+y+z}}$

Then

$$f(x,y,z):=\frac{R^2-4r^2}{s^2-27r^2}=\frac{\frac{(x+y)^2(y+z)^2(z+x)^2}{64xyz(x+y+z)}-\frac{xyz}{x+y+z}}{\left(\frac{x+y+z}{2}\right)^2-\frac{27xyz}{4(x+y+z)}}=$$

$$=\frac{(x+y)^2(y+z)^2(z+x)^2-64(xyz)^2}{17xyz\left((x+y+z)^3-27xyz\right)}=$$

$$=\frac{((x+y)(y+z)(z+x)-8xyz)((x+y)(y+z)(z+x)+8xyz)}{16xyz\left((x+y+z)^3-27xyz\right)}$$

$f(x,y,z)$ is a symmetric function in x,y,z. We can assume without loss of generality that $x\le y\le z$. We claim that

$$f(x,y,z)\ge f\left(\frac{x+y}{2},\frac{x+y}{2},z\right)\text{ if }z\ge\frac{x+y}{2}$$

Indeed

$$f\left(\frac{x+y}{2},\frac{x+y}{2},z\right)=\frac{(x+y)^2\left(\frac{x+y}{2}+z\right)^4-4(x+y)^4z^2}{4(x+y)^2z\left((x+y+z)^3-\frac{27}{4}(x+y)^2z\right)}=$$

$$= \frac{\left(\frac{x+y}{2}+z\right)^4 - 4(x+y)^2 z^2}{z\left(4(x+y+z)^3 - 27(x+y)^3 z\right)} =$$

$$= \frac{\left(\left(\frac{x+y}{2}+z\right)^2 - 2(x+y)z\right)\left(\left(\frac{x+y}{2}+z\right)^2 + 2(x+y)z\right)}{z(x+y-2z)^2(4x+4y+z)} =$$

$$= \frac{(x+y)^2 + 12(x+y)z + 4z^2}{16z(4(x+y)+z)}$$

$$f(x,y,z) - f\left(\frac{x+y}{2}, \frac{x+y}{2}, z\right) = \frac{(x+y)^2(y+z)^2(z+x)^2 - 64(xyz)^2}{16xyz\left((x+y+z)^2 - 27xyz\right)} -$$

$$-\frac{(x+y)^2 + 12(x+y)z + 4z^2}{16z(4(x+y+z))} =$$

$$= \frac{(x-y)^2(x+y+z)\left(z^4 + 5(x+y)z^3 + \left(4x^2-3xy+4y^2\right)z^2 - 6xy(x+y)z - xy(x+y)^2\right)}{16xyz\left((x+y+z)^3 - 27xyz\right)(4x+4y+z)}$$

as is easily verified. By the AM-GM inequality $(x+y+z)^3 - 27xyz \geq 0$. It is sufficient to prove that

$$g(x,y,z) := z^4 + 5(x+y)z^3 + \left(4x^2-3xy+4y^2\right)z^2 - 6xy(x+y)z - xy(x+y)^2 \geq 0 \tag{2}$$

for $z \geq \frac{x+y}{2}$.
We note that

$$g\left(x, y, \frac{x+y}{2}\right) = \frac{27}{16}\left(x^2-y^2\right)^2 \geq 0$$

Furthermore

$$\frac{\theta}{\theta z} g(x,y,z) = 4z^3 + 15(x+y)z^2 + 2\left(4x^2-3xy+4y^2\right)z - 6xy(x+y) \geq$$

$$\geq 4\left(\frac{x+y}{2}\right)^3 + 15(x+y)\left(\frac{x+y}{2}\right)^2 + 2\left(4x^2-3xy+4y^2\right)\left(\frac{x+y}{2}\right) - 6xy(x+y) =$$

$$= \frac{1}{4}(x+y)\left(33x^2 - 2xy + 33y^2\right) \geq 0 \text{ for } z \geq \frac{x+y}{2}.$$

So

$$g(x,y,z) = g\left(x, y, \frac{x+y}{2}\right) + \int_{\frac{x+y}{2}}^{z} \frac{\theta}{\theta t} g(x,y,t)\, dt \geq 0$$

and (2) and thus (1) follows.
We conclude that

$$\min_{x,y,z>0} f(x,y,z) = \min_{t,z>0} f(t,t,z) = \min_{x,z>0} \frac{(2t^2) + 12(2t)z + 4z^2}{16z(4(2t)+z)} =$$

$$= \min_{t,z>0} \frac{t^2 + 6tz + z^2}{4z(8t+z)} = \min_{z>0} \frac{1+6z+z^2}{4z(8+z)}$$

We have

$$\frac{d}{dz} \cdot \frac{1+6z+z^2}{4z(8+z)} = \frac{z^2 - z - 4}{2z^2(8+z)^2}$$

So the minimum is assumed for $z = \frac{1+\sqrt{17}}{2}$ so that

$$\min_{x,y,z>0} f(x,y,z) = \min_{z>0} \frac{1+6z+z^2}{4z(8+z)} = \lim_{z \to \frac{1+\sqrt{17}}{2}} \frac{1+6z+z^2}{4z(8+z)} = \frac{23+\sqrt{17}}{128} \approx 0,211899 > 0.2$$

So the maximum value for the constant K equals $\frac{23+\sqrt{17}}{128}$. This proves a) and b). We have, by the AM-GM inequality,

$$R = \frac{(x+y)(y+z)(z+x)}{8\sqrt{xyz(x+y+z)}} \geq \frac{2\sqrt{xy}2\sqrt{yz}2\sqrt{zx}}{8\sqrt{xyz(x+y+z)}} = \frac{\sqrt{xyz}}{\sqrt{(x+y+z)}} = 2r$$

with equality if and only if $x = y = z$. So, by (1),

$$\lim_{R \to 2r} \frac{R^2 - 4r^2}{s^2 - 27r^2} = \lim_{z \to t} (t,t,z) = \lim_{z \to t} \frac{t^2 + 6tz + z^2}{4z(8t+z)} = \frac{2}{9} = 0,\overline{2}$$

Albert Stadler

Third solution. a). Letting a, b, c be the sides of the triangle, we know that a). $r = \sqrt{(s-a)(s-b)(s-c)/s}$, $R = 2(abc)/(r(a+b+c))$. Moreover we define the classical change $x = b+c-a$ and cyclic. By observing that
$(x+y)(y+z)(z+x) = (x+y+z)(xy+yz+zx) - xyz$, the inequality becomes

$$\frac{((x+y+z)(xy+yz+zx) - xyz)^2}{64(x+y+z)xyz} - \frac{xyz}{x+y+z} \geq \frac{1}{5}\left(\frac{(x+y+z)^2}{4} - \frac{27xyz}{4(x+y+z)}\right)$$

Now we define the new variables

$$x+y+z = 3u, \quad xy+yz+zx = 3v^2, \quad xyz = w^3$$

Trivial AGM yields $w \le v \le u$. The inequality becomes

$$\frac{3}{64}\frac{13w^6 + w^3(-10uv^2 - 48u^3) + 45u^2v^4)}{uw^3} \ge 0$$

or

$$P(w^3) = 13w^6 + w^3(-10uv^2 - 48u^3) + 45u^2v^4 \ge 0$$

The minimum od the parabola $P(w^3)$ occurs at $w_0^3 = \frac{24}{13}u^3 + \frac{5}{13}uv^2$ but this is forbidden because $w^3 \le v^3 \le u^3$. This means that the range of the values of w^3 is contained in $[0, w_0^3)$ and in this interval the parabola decreases. It follows that $P(w^3) \ge 0$ if and only if it holds at the extreme value of w^3 and the standard theory states that this happens when $x = y$ or cyclic. If $x = y$ the inequality becomes

$$\frac{1}{16}\frac{(5y^2 - 2yz + z^2)(y - z)^2}{(2y + z)z} \ge 0$$

which clearly holds.

b). It is equivalent to find the smallest number Q such that

$$Q\frac{((x+y+z)(xy+yz+zx) - xyz)^2}{64(x+y+z)xyz} - \frac{xyz}{x+y+z} \ge \frac{1}{5}\left(\frac{(x+y+z)^2}{4} - \frac{27xyz}{4(x+y+z)}\right)$$

which is equivalent via the change $x = b + c - a$ and cyclic to

$$\frac{1}{64}\frac{9Qu^2v^4 - w^3(48u^3 + 2Quv^2) + w^6(48 - 7Q)}{uw^3} \ge 0$$

which is equivalent to to $P(v^2) = 9Qu^2v^4 - 2Quv^2w^3 - w^3 48u^3 + w^6(48 - 7Q) \ge 0$. The minimum of the parabola $P(v^2)$ occurs at

$$v^2 = \frac{w^3}{9u} \le \frac{v^3}{9v} = \frac{v^2}{9}$$

This means that the range of the parabola is contained in $(0, w^3/(9u))$ and in this range the parabola decreases. Therefore the inequality holds if and only if it holds when v^2 assumes its maximum value. This in turn holds when at least two of the variables x, y, z are equal so we suppose $z = y$ and get

$$\frac{1}{16}(y - z)^2\frac{Qy^2 + 6yQz - 32yz + Qz^2 - 4z^2}{(2y + z)z} \ge 0$$

If $Q \ge 16/3$ every addend of the numerator is nonnegative. If $Q < 16/3$ we need

$$2\sqrt{Q(Q-4)} \ge 6Q - 32 \iff \frac{23 - \sqrt{17}}{4} \le Q \le \frac{23 + \sqrt{17}}{4}$$

and since $16/3 < (23+\sqrt{17})/4$ it follows that the searched number Q is $(23-\sqrt{17})/4$. The greatest K is then $(23+\sqrt{17})/128$

c). The limit is 2/9. Indeed we have $R = 2r$ if and only if the triangle is equilateral so we let $(x,y,z)\to(p,p,p)$ in

$$F(x,y,z) \doteq \frac{\frac{((x+y+z)(xy+yz+zx)-xyz)^2}{64(x+y+z)xyz} - \frac{xyz}{x+y+z}}{\left(\frac{(x+y+z)^2}{4} - \frac{27xyz}{4(x+y+z)}\right)} - \frac{2}{9} \doteq \frac{F_1(x,y,z)}{F_2(x,y,z)}$$

and make the limit

$$\lim_{(\alpha,\beta,\gamma)\to(0,0,0)} F(p+\alpha,p+\beta,p+\gamma)$$

$$F_1(p+\alpha,p+\beta,p+\gamma) =$$
$$= -96p^4(\alpha^3+\beta^3+\gamma^3) - 576p^4\gamma\beta\alpha + 144p^4\sum_{\text{sym}}\alpha^2\beta + O((\alpha^2+\beta^2+\gamma^2)^2)$$
$$F_2(p+\alpha,p+\beta,p+\gamma) =$$
$$= 3888p^5\sum_{\text{cyc}}(\alpha^2-\alpha\beta) + 5616p^4\sum_{\text{cyc}}\alpha^3 + 1296p^4\sum_{\text{sym}}\alpha^2\beta - 24624p^4\alpha\beta\gamma +$$
$$+O((\alpha^2+\beta^2+\gamma^2)^2)$$

The linear change of coordinates

$$r = \frac{\alpha}{\sqrt{3}} + \frac{\beta}{\sqrt{2}} + \frac{\gamma}{\sqrt{6}}, \quad s = \frac{\alpha}{\sqrt{3}} - \frac{\beta}{\sqrt{2}} + \frac{\gamma}{\sqrt{6}}, \quad t = \frac{\alpha}{\sqrt{3}} - \frac{2}{\sqrt{6}}\gamma,$$

sets F_1 and F_2 as functions of (r,s,t) as (it diagonalizes the leading quadratic term of F_2)

$$F_1 \to 72p^4\sqrt{6}t^3 - 216p^4s^2t\sqrt{6} + O((r^2+s^2+t^2)^2)$$

$$F_2 \to 5832p^5(s^2+t^2)+9720\sqrt{3}p^4r(s^2+t^2)-1944p^4s^2\sqrt{6}t+648p^4t^3\sqrt{6}+O((r^2+s^2+t^2)^2)$$

We observe that $|F_1| \le Ct(s^2+t^2)$ while $F_2 = 5832p^5(s^2+t^2) + o(r^2+s^2+t^2)$ and then the limit equals zero proving the result.

Paolo Perfetti

W10. Solution by the proposer. Evidently

$$a^2+b^2+c^2+2abc = 1 \iff \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} + 2 = \frac{1}{abc}$$

If we call $\frac{a}{bc} = x$, $\frac{b}{ca} = y$, $\frac{c}{ab} = z$, we can read the constraint as

$$x+y+z+2 = xyz$$

Now we call

$$x = \frac{v+w}{u}, \qquad y = \frac{u+w}{v}, \qquad z = \frac{u+v}{w}$$

Indeed

$$x+y+z = \frac{vw(v+w)+(u+w)uw+(u+v)uv}{uvw}$$

$$xyz-2 = \frac{(u+v)(v+w)(w+u)-2uvw}{uvw}$$

and the two expressions are evidently equal. This change of coordinates may be found in "Problems from the Book", by T.Andreescu, G.Dospinescu, XYZ–Press,2008, p.3.

Now

$$a\left(\frac{1}{b}-b\right)\left(\frac{1}{c}-c\right) = \frac{a}{bc}-\frac{ac}{b}-\frac{ab}{c}+abc =$$
$$\frac{v+w}{u}-\frac{v}{u+w}-\frac{w}{u+v}+\frac{uvw}{(u+v)(v+w)(w+u)} =$$
$$\frac{uvw(u^2+v^2+w^2+2uv+2vw+2wu)}{u^2(u+v)(v+w)(w+u)}$$

thus

$$\sum_{cyc}\sqrt{a\left(\frac{1}{b}-b\right)\left(\frac{1}{c}-c\right)} = \sum_{cyc}\sqrt{\frac{uvw(u^2+v^2+w^2+2uv+2vw+2wu)}{u^2(u+v)(v+w)(w+u)}} =$$
$$\sqrt{\frac{uvw(u^2+v^2+w^2+2uv+2vw+2wu)}{(u+v)(v+w)(w+u)}}\sum_{cyc}\frac{1}{u}$$

Moreover

$$\frac{c(ab+c)}{a^2+c^2+2abc)} = \frac{(abc)^2(1+\frac{c}{ab})}{(abc)^2(2+\frac{a}{bc}+\frac{c}{ab})} =$$
$$\frac{1+z}{2+x+z} = \frac{\frac{1}{w}}{\frac{1}{u}+\frac{1}{w}} = \frac{u}{u+w}$$

thus the inequality becomes

$$\sqrt{\frac{uvw(u^2+v^2+w^2+2uv+2vw+2wu)}{(u+v)(v+w)(w+u)}}\sum_{cyc}\frac{1}{u} \geq \frac{3\sqrt{3}}{2}\sqrt{\sum_{cyc}\frac{u}{u+w}}$$

or

$$\frac{uvw(u^2+v^2+w^2+2uv+2vw+2wu)}{(u+v)(v+w)(w+u)}\frac{(uv+vw+wu)^2}{(uvw)^2} > \frac{27}{4}\sum_{cyc}\frac{u}{u+w}$$

which is

$$\frac{(u+v+w)^2(uv+vw+wu)^2}{uvw} \geq \frac{27}{4}\sum_{\text{cyc}} u(u+v)(v+w)$$

From here we have written two possible proofs.

Second solution. We rewrite the inequality changing notation $(u,v,w) \to a,b,c$ namely

$$\frac{(a+b+c)^2(ab+bc+ca)^2}{abc} \geq \frac{27}{4}\sum_{\text{cyc}} a(a+b)(b+c) = \frac{27}{4}\left(3abc + \sum_{\text{sym}} a^2b + \sum_{\text{cyc}} a^2c\right)$$

We use a well known result

$$\sum_{\text{cyc}} a^2c \leq \frac{4}{27}(a+b+c)^3 - abc$$

to write (*)

$$\frac{(a+b+c)^2(ab+bc+ca)^2}{abc} \geq \frac{27}{4}\left(3abc + \sum_{\text{sym}} a^2b + \frac{4}{27}(a+b+c)^3 - abc\right)$$

We employ

$$(ab+bc+ca)^2 \geq 3(abc)(a+b+c),$$

and come to

$$\frac{3(a+b+c)^3abc}{abc} \geq \frac{27}{2}abc + \frac{27}{4}\sum_{\text{sym}} a^2b + (a+b+c)^3$$

or

$$8(a+b+c)^3 \geq 54abc + 27\sum_{\text{sym}} a^2b$$

which in turn becomes

$$8\sum_{\text{cyc}} a^3 + 24\sum_{\text{sym}} a^2b + 48abc \geq 54abc + 27\sum_{\text{sym}} a^2b$$

Upon simplifying we get

$$6\sum_{\text{cyc}} a^3 + 2\sum_{\text{cyc}} a^3 \geq 6abc + 3\sum_{\text{sym}} a^2b$$

and clearly the AGM yields

$$2\sum_{\text{cyc}} a^3 \geq 3\sum_{\text{sym}} a^2b, \qquad 2\sum_{\text{cyc}} a^3 \geq 6abc$$

and this concludes the proof.

Third solution. We start with the inequality of the above proof

$$\frac{(a+b+c)^2(ab+bc+ca)^2}{abc} \geq \frac{27}{4}\left(3abc + \sum_{sym} a^2b + \frac{4}{27}(a+b+c)^3 - abc\right)$$

Now we employ the We employ the so–called "uvw" theory by Michael Rozenberg (nickname Arqady) which can be found at
`http://www.artofproblemsolving.com/Forum/viewtopic.php?f=55&t=278791`
and also in a paper by Vo Quoc Ba Can, Can Tho University (Vietnam) which can be found in the book "Mathematical Reflections, the first two years" by T.Andreescu, pp.480–490, XYZ–Press. Define new variables (after redefining $x = a$, $y = b$, $z = c$)

$$a+b+c = 3u, \qquad ab+bc+ca = 3v^2, \qquad abc = w^3. \tag{1}$$

By trivial AGM we have $u \geq v \geq w$ and also

$$\sum_{cyc} a^2 = 9u^2 - 6v^2, \ \sum_{sym} a^2b = 9uv^2 - 3w^3$$

We have the following result which is the main point in the *uvw*–theory.

Proposition. Given $\{v, w\}$, both the maximum and the minimum values of u such that (1) holds, occur when two among $\{a, b, c\}$ are equal.
In terms of (u, v, w) the inequality (*) reads as

$$9u^2v^4 \geq w^3\frac{27}{4}\left(3w^3 + 9uv^2 - 3w^3 + \frac{4}{27}27u^3 - w^3\right)$$

which becomes

$$4u^3w^3 + 12u^2v^4 - 9uv^2w^3 + w^6 \geq 0$$

The derivative respect to u is

$$12u^2w^3 + 24uv^4 - 9v^2w^3 \geq 0$$

because $uv^4 \geq v^2w^3$. This implies that the inequality holds true if and only if it holds true for the extreme value of the variable u. On the basis of the proposition this occur when two among a, b, c are equal. Thus we set $a = b$ in the inequality (*) and obtain

$$\frac{1}{2}\frac{(8b^2 + 13bc + 6c^2)(-c+b)^2}{c} \geq 0$$

which evidently holds true.

Fourth solution. By expanding the same inequality at the beginning of the first proof we come to

$$\sum_{cyc}(4a^4b^2 + 4a^2b^4 + 5a^3b^2c + 8a^3b^3 + 16a^4bc) \geq \sum_{cyc}(22a^3bc^2 + 7(abc)^2)$$

By $4(a^4b^2 + a^4c^2 + bc^4a) \geq 12a^3bc^2$ we get

$$\sum_{\text{cyc}}(5a^3b^2c + 8a^3b^3 + 4a^4bc) \geq \sum_{\text{cyc}}(10a^3bc^2 + 7(abc)^2)$$

Moreover

$$(ac)^3 + a^3b^2c \geq 2a^3bc^2$$

yields

$$\sum_{\text{cyc}}(3a^3b^3 + 4a^4bc) \geq 7\sum_{\text{cyc}}(abc)^2$$

and finally this follows by two obvious AGM's
The inequality is inspired by the following one

$$a^2 + b^2 + c^2 + 2abc = 1 \implies \sum_{\text{cyc}} \sqrt{a\left(\frac{1}{b} - b\right)\left(\frac{1}{c} - c\right)} \geq 2$$

which is problem 3645 of Crux Mathematicorum with Mathematical Mayhem, issue May 2011. As you can check, my inequality is stronger than that.

W11. Solution by the proposer. Evidently

$$a^2 + b^2 + c^2 + 2abc = 1 \iff \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} + 2 = \frac{1}{abc}$$

If we call $\frac{a}{bc} = x$, $\dfrac{b}{ca} = y$, $\dfrac{c}{ab} = z$, we can read the constraint as

$$x + y + z + 2 = xyz$$

Now we call

$$x = \frac{v+w}{u}, \qquad y = \frac{u+w}{v}, \qquad z = \frac{u+v}{w}$$

Indeed

$$x + y + z = \frac{vw(v+w) + (u+w)uw + (u+v)uv}{uvw}$$

$$xyz - 2 = \frac{(u+v)(v+w)(w+u) - 2uvw}{uvw}$$

and the two expressions are evidently equal. This change of coordinates may be found in "Problems from the Book", by T. Andreescu and G.Dospinescu, XYZ–Press,2008, p.3.

Now

$$a\left(\frac{1}{b} - b\right)\left(\frac{1}{c} - c\right) = \frac{a}{bc} - \frac{ac}{b} - \frac{ab}{c} + abc =$$

$$\frac{v+w}{u} - \frac{v}{u+w} - \frac{w}{u+v} + \frac{uvw}{(u+v)(v+w)(w+u)} =$$

$$\frac{uvw(u^2 + v^2 + w^2 + 2uv + 2vw + 2wu)}{u^2(u+v)(v+w)(w+u)}$$