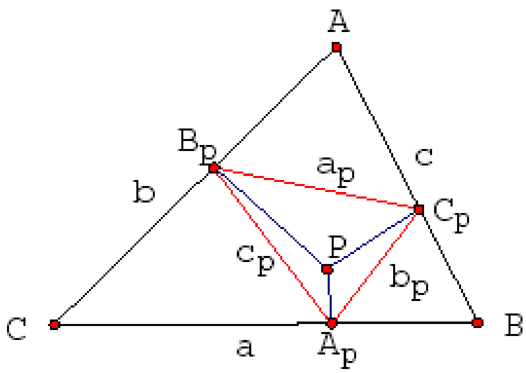
Problem with a solution proposed by Arkady Alt , San Jose , California, USA.
One inequality with vanished variables in a triangle.

Let a,b,c be side lengths of a triangle ABC and x,y,z be non-negative real numbers such that $x+y+z=1$ and let R be circumradius of this triangle. Prove that

$$a^2yz+b^2zx+c^2xy\le R^2$$

Solution.



Let P be point in $\triangle ABC$ with barycentric coordinates $(p_a,p_b,p_c)=(x,y,z)$.
Let $R_a:=PA,R_b=PB,R_c:=PC$ and A_p,B_p,C_p be foots of perpendiculars from P to sides BC,CA,AB respectively. Also we denote via $d_a:=PA_p$, $d_b:=PB_p,d_c:=PC_p$ and $a_p:=B_pC_p,b_p:=C_pA_p,c_p:=A_pB_p$ (side lengths of pedal triangle $A_pB_pC_p$). Let $F:=[ABC]$ and $F_a:=[PBC],F_b:=[PCA],F_c:=[PAB]$.
Then $F_a=\dfrac{ad_a}{2},F_b=\dfrac{bd_b}{2},F_c=\dfrac{cd_c}{2}$ and $p_a=\dfrac{F_a}{F}=\dfrac{ad_a}{2F},p_b=\dfrac{F_b}{F}=\dfrac{bd_b}{2F}$, $p_c=\dfrac{F_c}{F}=\dfrac{cd_c}{2F}$.
Since $\angle B_pPC_p=180^\circ-A$ then, by Cos-Theorem, $a_p^2=d_b^2+d_c^2+2d_bd_c\cos A$ and $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$.
Using $d_b=\dfrac{2p_bF}{b}$, $d_c=\dfrac{2p_cF}{c}$ and $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$ we obtain:

$$a_p^2=\frac{4p_b^2F^2}{b^2}+\frac{4p_c^2F^2}{c^2}+\frac{4p_bp_cF^2}{bc}\cdot\frac{b^2+c^2-a^2}{bc}=$$
$$\frac{4F^2}{b^2c^2}(p_b^2c^2+p_c^2b^2+p_bp_c(b^2+c^2-a^2))=$$
$$\frac{4F^2}{b^2c^2}(p_b(1-p_c-p_a)c^2+p_c(1-p_a-p_b)b^2+p_bp_c(b^2+c^2-a^2))=$$
$$\frac{4F^2}{b^2c^2}(p_bc^2+p_cb^2-p_bp_ca^2-p_cp_ab^2-p_ap_bc^2).$$

Since $abc=4FR$ then $\dfrac{4F^2}{b^2c^2}=\dfrac{4a^2F^2}{a^2b^2c^2}=\dfrac{a^2F^2}{4R^2}$ and, therefore,

(1) $$a_p^2=\frac{a^2F^2}{4R^2}(p_bc^2+p_cb^2-p_bp_ca^2-p_cp_ab^2-p_ap_bc^2).$$

Also, since quadrilateral AB_pPC_p cyclic with diameter R_a, by Sine Theorem we obtain $a_p=R_a\sin A=R_a\cdot\dfrac{a}{2R}=\dfrac{aR_a}{2R}$.
By substitution $a_p=\dfrac{aR_a}{2R}$ in (**1**) we obtain barycentric representation for R_a^2 :

(2) $\boxed{R_a^2 = p_b c^2 + p_c b^2 - p_b p_c a^2 - p_c p_a b^2 - p_a p_b c^2}$.

Since $p_a = \dfrac{ad_a}{2F}, p_b = \dfrac{bd_b}{2F}, p_c = \dfrac{cd_c}{2F}$ then $p_b c^2 + p_c b^2 = \dfrac{bc^2 d_b}{2F} + \dfrac{b^2 c d_c}{2F} = \dfrac{bc}{2F}(cd_b + bcd_c)$ and applying inequality $aR_a \geq cd_b + bd_c$ we obtain

$$p_b c^2 + p_c b^2 \leq \frac{abcR_a}{2F} = 2RR_a .$$

Since $p_b = \dfrac{bd_b}{2F}$ and $p_c = \dfrac{cd_c}{2F}$ then $p_b p_c a^2 = \dfrac{a^2 b^2 c^2}{4F^2} \cdot \dfrac{d_b d_c}{bc} = 4R^2 \cdot \dfrac{d_b d_c}{bc}$.

Thus, $R_a^2 = p_b c^2 + p_c b^2 - p_b p_c a^2 - p_c p_a b^2 - p_a p_b c^2 \leq 2RR_a - 4R^2 \sum_{cyclic} \dfrac{d_b d_c}{bc} \Leftrightarrow$

$(R - R_a)^2 \leq R^2 - 4R^2 \sum_{cyc} \dfrac{d_b d_c}{bc} \Rightarrow \sum_{cyc} \dfrac{d_b d_c}{bc} \leq \dfrac{1}{4}$ and, therefore,

$a^2 yz + b^2 zx + c^2 xy = \sum_{cyc} p_b p_c a^2 = \sum_{cyc} \dfrac{bd_b}{2F} \cdot \dfrac{cd_c}{2F} a^2 = \sum_{cyc} \dfrac{a^2 bcd_b d_c}{4F^2} =$

$\dfrac{a^2 b^2 c^2}{4F^2} \sum_{cyc} \dfrac{d_b d_c}{bc} = 4R^2 \sum_{cyc} \dfrac{d_b d_c}{bc} \leq 4R^2 \cdot \dfrac{1}{4} = R^2$.