$$|f(I_n)| = 1 + \left[\frac{3m}{2}\right] = 1 + \left[\frac{6m}{4}\right] = 1 + \left[\frac{3n}{4}\right].$$

For $n = 2m+1$ we have

$$|f(I_n)| = 1 + \left[\frac{3m^2+3m+1}{2m+1}\right] = 1 + \left[\frac{\frac{12m^2+12m+4}{2m+1}}{4}\right] =$$

$$= 1 + \left[\frac{\frac{3(2m+1)^2+1}{2m+1}}{4}\right] = 1 + \left[\frac{\left[3(2m+1) + \frac{1}{2m+1}\right]}{4}\right] =$$

$$= 1 + \left[\frac{3(2m+1)}{4}\right] = 1 + \left[\frac{3n}{4}\right].$$

So,

$$|f(I_n)| = 1 + \left[\frac{3n}{4}\right].$$

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W17. (Solution by the proposer.) First we solve the recurrence

$$t_{k+2} - t_{k+1} - t_k = f_k, k \in \mathbb{N}, \text{where} f_k, k \in \mathbb{N} \text{ are Fibonacci numbers.} \quad (1)$$

For any sequence $(t_n)_{n\geq 0}$ we set $F(t_n) := t_{n+2} - t_{n+1} - t_n$ and call F Fibonacci Operator.
Easy to see that F is linear operator, that is
$F(at'_n + bt''_n) = aF(t'_n) + bF(t''_n)$ for any $a, b \in \mathbb{R}$.
Noting that

$$F(nf_n) = (n+2) f_{n+2} - (n+1) f_{n+1} - nf_n =$$

$$= 2f_{n+2} - f_{n+1} = f_{n+1} + 2f_n.$$

and

$$F(nf_{n-1}) = (n+2) f_{n+1} - (n+1) f_n - nf_{n-1} =$$

$$= 2f_{n+2} - f_{n+1} = 2f_{n+1} - f_n$$

we obtain

$$2F(nf_n) - F(nf_{n-1}) = 5f_n \iff f_n = F\left(\frac{2nf_n - nf_{n-1}}{5}\right), n \in N.$$

Hence,

$$(1) \quad \iff F(t_n) = F\left(\frac{2nf_n - nf_{n-1}}{5}\right) \iff F\left(t_n - \frac{2nf_n - nf_{n-1}}{5}\right) =$$

$$= 0 \iff t_n = \frac{2nf_n - nf_{n-1}}{5} + af_n + bf_{n-1} n \in \mathbb{N}$$

(* because for any solution s_n of equation $F(s_n) = 0$ there is real constants a and b such that $s_n = af_n + bf_{n-1}, n \in \mathbb{N}$).
We determine constant a, b such that

$$\begin{cases} t_1 = x_1 \\ t_2 = x_2 \end{cases} \iff \begin{cases} \dfrac{2f_1 - f_0}{5} + af_1 + bf_0 = x_1 \\ \dfrac{2 \cdot 2f_2 - 2f_1}{5} + af_2 + bf_1 = x_2 \end{cases} \iff$$

$$\begin{cases} \dfrac{2}{5} + a = x_1 \\ \dfrac{2}{5} + a + b = x_2 \end{cases} \iff \begin{cases} a = x_1 - \dfrac{2}{5} \\ b = x_2 - x_1 \end{cases}.$$

Then for

$$t_n = \frac{2nf_n - nf_{n-1}}{5} + \left(x_1 - \frac{2}{5}\right) f_n + (x_2 - x_1) f_{n-1} =$$

$$= \frac{2(n-1) f_n - nf_{n-1}}{5} + x_1 f_n + (x_2 - x_1) f_{n-1} =$$

$$= \frac{2(n-1) f_n - nf_{n-1}}{5} + x_1 f_{n-2} + x_2 f_{n-1}$$

and since $t_1 = x_1, t_2 = x_2$ then $t_k = x_k, k = 1, 2, ..., n$ because
$F(t_k) = f_k, k = 1, 2, ..., n-2$ and in original system
$F(x_k) = f_k, k = 1, 2, .., n-2$.(Since $t_1 = x_1, t_2 = x_2$ and for any
$k = 1, 2, .., n-2$ we have

$$x_{k+1} = t_{k+1}, x_k = t_k \implies x_{k+2} = x_{k+1} + x_k + f_k = t_{k+1} + t_k + f_k = t_{k+2}$$

then by Math Induction $t_k = x_k, k = 1, 2, ..., n)$. Thus,

$$x_k = \frac{2(k-1) f_k - k f_{k-1}}{5} + x_1 f_k + (x_2 - x_1) f_{k-1}, k = 1, 2, .., n$$

and

$$\begin{cases} x_1 - x_n - x_{n-1} = f_{n-1} \\ x_2 - x_1 - x_n = f_n \end{cases}.$$

Noting that $x_1 = x_n + x_{n-1} + f_{n-1} = t_n + t_{n-1} + f_{n-1} = t_{n+1}$ and $x_2 = x_1 + x_n + f_n = t_{n+1} + t_n + f_n = t_{n+2}$ we obtain

$$\begin{cases} x_1 = t_{n+1} \\ x_2 = t_{n+2} \end{cases} \iff$$

$$\iff \begin{cases} x_1 = \dfrac{2n f_{n+1} - (n+1) f_n}{5} + x_1 f_{n-1} + x_2 f_n \\ x_2 = \dfrac{2(n+1) f_{n+2} - (n+2) f_{n+1}}{5} + x_1 f_n + x_2 f_{n+1} \end{cases} \iff$$

$$\begin{cases} x_1 (1 - f_{n-1}) - x_2 f_n = \dfrac{2n f_{n+1} - (n+1) f_n}{5} \\ -x_1 f_n + x_2 (1 - f_{n+1}) = \dfrac{2(n+1) f_{n+2} - (n+2) f_{n+1}}{5} \end{cases}. \qquad (2)$$

Checked! Since $f_{n-1} f_{n+1} - f_n^2 = (-1)^n$ (Cassiny Identity) then

$$\det \begin{pmatrix} 1 - f_{n-1} & -f_n \\ -f_n & 1 - f_{n+1} \end{pmatrix} =$$

$$= (1 - f_{n-1})(1 - f_{n+1}) - f_n^2 = 1 + (-1)^n - f_{n-1} - f_{n+1} \neq 0$$

for any $n \in \mathbb{N}$ and we obtain :

1. By elimination x_2 :

$$x_1 (1 + (-1)^n - f_{n-1} - f_{n+1}) =$$

$$= \frac{(2n f_{n+1} - (n+1) f_n)(1 - f_{n+1})}{5} + \frac{(2(n+1) f_{n+2} - (n+2) f_{n+1}) f_n}{5} =$$

$$= \frac{\left(2f_{n+1} - f_n + 2\left(f_n f_{n+2} - f_{n+1}^2\right)\right) n + f_n \left(2f_{n+2} - f_{n+1} - 1\right)}{5} =$$

$$= \frac{\left(f_{n+1} + f_{n-1} + 2\left(-1\right)^{n+1}\right) n + f_n \left(f_{n+2} + f_n - 1\right)}{5} \Longleftrightarrow$$

$$x_1 = \frac{\left(f_{n+1} + f_{n-1} + 2\left(-1\right)^{n+1}\right) n + f_n \left(f_{n+2} + f_n - 1\right)}{5\left(1 + (-1)^n - f_{n-1} - f_{n+1}\right)}$$

and
Checked!

2. By elimination x_1 :

$$x_2\left(1 + (-1)^n - f_{n-1} - f_{n+1}\right) = \frac{\left(2nf_{n+1} - (n+1)\, f_n\right) f_n}{5} +$$

$$+ \frac{\left(\left(2\,(n+1)\, f_{n+2} - (n+2)\, f_{n+1}\right) f_n\right)\left(1 - f_{n-1}\right)}{5} =$$

$$= \frac{f_n\left(f_{n+1} - f_n + 2f_{n+2} + f_{n-1}f_{n+1} - 2f_{n-1}f_{n+2}\right) n - f_n\left(f_n + 2f_{n+1} - 2f_{n+2} - 2f_{n-1}\right.}{5}$$

$$= \frac{f_n}{5}\left(\left(f_{n+1} - f_n + 2f_{n+2} + f_{n-1}f_{n+1} - 2f_{n-1}f_{n+2}\right) n - \right.$$

$$\left. - \left(f_n + 2f_{n+1} - 2f_{n+2} - 2f_{n-1}f_{n+1} + 2f_{n-1}f_{n+2}\right)\right).$$

Since

$$f_{n+1} - f_n + 2f_{n+2} + f_{n-1}f_{n+1} - 2f_{n-1}f_{n+2} = 2f_{n+1} + f_{n+2} - f_{n-1}\left(2f_{n+2} - f_{n+1}\right) =$$

$$= 2f_{n+1} + f_{n+2} - f_{n-1}\left(2f_n + f_{n+1}\right)$$

and

$$f_n + 2f_{n+1} - 2f_{n+2} - 2f_{n-1}f_{n+1} + 2f_{n-1}f_{n+2} = -f_n + 2f_{n-1}f_n = f_n\left(2f_{n-1} - 1\right)$$

then

$$x_2 = \frac{f_n\left(2f_{n+1}+f_{n+2}-f_{n-1}\left(2f_n+f_{n+1}\right)\right)n - f_n^2\left(2f_{n-1}-1\right)}{5\left(1+(-1)^n-f_{n-1}-f_{n+1}\right)}.$$

Thus, solution of the system can be calculated by formulas:

$$x_1 = \frac{\left(f_{n+1}+f_{n-1}+2(-1)^{n+1}\right)n + f_n\left(f_{n+2}+f_n-1\right)}{5\left(1+(-1)^n-f_{n-1}-f_{n+1}\right)},$$

$$x_2 = \frac{f_n\left(2f_{n+1}+f_{n+2}-f_{n-1}\left(2f_n+f_{n+1}\right)\right)n - f_n^2\left(2f_{n-1}-1\right)}{5\left(1+(-1)^n-f_{n-1}-f_{n+1}\right)}$$

and

$$x_k = \frac{2(k-1)f_k - kf_{k-1}}{5} + x_1 f_k + (x_2 - x_1) f_{k-1}, k = 3, .., n.$$

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Second solution. For every $k \in [1, n-1]$, by suming up every line of the system which contains x_k, we have :

$$-x_k - (n-2)x_{k+1} = S_k,$$

and

$$-x_n - (n-2)x_1 = S_1,$$

where $S_1, ...S_k$ are integers satisfying the relation :

$$\sum_{k=1}^{n} S_k = (n-1)\sum_{k=1}^{n} F_k = (n-1)(F_{n-2}-1).$$

Now, let $X := \sum_{k=1}^{n} x_k$. By suming up all the above equations, we have :

$$-(n-1)X = (n-1)(F_{n+2}-1) \Leftrightarrow X = -(F_{n+2}-1).$$

But on the other side, by suming up all the equation of the initial system, we have :

$$-(n-2)X = F_{n+2} - 1 \Leftrightarrow X = -\frac{F_{n+2}-1}{n-2}.$$

Hence, the system of equations can be solved if and only if $F_{n+2} - 1 = \frac{F_{n+2}-1}{n-2}$, which means that $n = 3$.