As example we apply obtained formula to determine $\sigma_{3,3}(n)$:

$$\sigma_{3,3}(n) = \sum_{i=1}^{3} (-1)^{3-i} \binom{3}{i} \left((n+1)^i - n^i\right) s_{k+m-i}(n) = \left((n+1)^3 - n^3\right) s_3(n) -$$

$$-3\left((n+1)^2 - n^2\right) s_4(n) + 3\left((n+1) - n\right) s_5(n) =$$

$$= \left(3n^2 + 3n + 1\right) s_3(n) - 3(2n+1) s_4(n) + 3s_5(n).$$

Since

$$s_4(n) = \frac{n(n+1)(2n+1)\left(3n^2+3n-1\right)}{30}, s_5(n) = \frac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12}$$

then

$$\sigma_{3,3}(n) = \frac{n^2(n+1)^2\left(3n^2+3n+1\right)}{4} - \frac{n(n+1)(2n+1)^2\left(3n^2+3n-1\right)}{10} +$$

$$+\frac{n^2(n+1)^2\left(2n^2+2n-1\right)}{4} = \frac{1}{20} n(n+1)\left(n^2+1\right)\left(n^2+2n+2\right).$$

Arkady Alt

W16. (Solution by the proposer.) Let $I_n := \{1, 2, ..., n\}$ and $f(k) := \left[\frac{k^2}{n}\right]$,for any $k \in I_n$. We have to dtermine $|f(I_n)|$.
Consider two cases.

Case1. n is even, that is $n = 2m$.
Lemma. For any $k \in I_m$ holds inequality

$$f(k+1) - f(k) \leq 1.$$

Proof. First we consider $k \in I_{m-1}$. Let $1 \leq k \leq m - 1$then

$$f(k+1) = \left[\frac{k^2+2k+1}{2m}\right]$$

and

$$1+f(k)=\left[\frac{k^2+2m}{2m}\right].$$

Note that $k\le m-1$ yields

$$f(k+1)\le\left[\frac{k^2+2(m-1)+1}{2m}\right]=\left[\frac{k^2+2m-1}{2m}\right]\le\left[\frac{k^2+2m}{2m}\right]=1+f(k).$$

Also for $k=m$ we have

$$f(m+1)=\left[\frac{m^2+2m+1}{2m}\right]=\left[\frac{\frac{m^2+2m+1}{2}}{m}\right]=$$

$$\left[\frac{\left[\frac{m^2+2m+1}{m}\right]}{2}\right]=\left[\frac{m+2+\left[\frac{1}{m}\right]}{2}\right]=1+\left[\frac{m}{2}\right]=1+f(m)$$

Corollary.

$$f(I_m)=\left\{0,1,2,...,\left[\frac{m}{2}\right]\right\}.$$

Proof. Note that $f(k)$ isn't decreasing, that is

$$f(k+1)-f(k)=\left[\frac{(k+1)^2}{n}\right]-\left[\frac{k^2}{n}\right]\ge 0.$$

Also,

$$f(m)=\left[\frac{m^2}{2m}\right]=\left[\frac{m}{2}\right]$$

and

$$f(1)=\left[\frac{1}{2m}\right]=0.$$

Suppose that there is $i\in I_{m/2}$ for which $f^{-1}(i)=\varnothing$. Obvious that $1\le i<\left[\frac{m}{2}\right]$.

Let $k_* := \{k \mid k \in I_m \text{ and } f(k) < i\}$.
Then

$$f(k_*) < i < f(k_* + 1) \implies f(k_* + 1) - f(k_*) > 1,$$

that is contradiction to Lemma.
Now note that $f(k)$ is strictly increasing in $k \in \{m+1, m+2, .., 2m\}$.
Indeed, since for any $k \in I_m$ we have

$$f(m+k) = \left[\frac{(m+k)^2}{2m}\right] = \left[\frac{m^2 + 2mk + k^2}{2m}\right] =$$

$$= k + \left[\frac{m^2 + k^2}{2m}\right]$$

then

$$f(m + (k+1)) = k + 1 + \left[\frac{m^2 + (k+1)^2}{2m}\right] >$$

$$> k + \left[\frac{m^2 + k^2}{2m}\right] = f(m+k)$$

for any $k \in I_{m-1}$.
Hence, $|f(I_{2m} \setminus I_m)| = m$ and since $|f(I_m)| = \left[\frac{m}{2}\right] + 1$ then

$$|f(I_{2m})| = m + \left[\frac{m}{2}\right] + 1 = \left[\frac{3m+2}{2}\right].$$

Case 2. n is odd, that is $n = 2m + 1$.
Then as above we will prove divide this case on two parts.
First we consider f on I_{m+1}.
For any $k \in I_m$ we have

$$f(k+1) = \left[\frac{k^2 + 2k + 1}{2m+1}\right] \le \left[\frac{k^2 + 2m + 1}{2m+1}\right] =$$

$$= 1 + \left[\frac{k^2}{2m+1}\right] = 1 + f(k+1)$$

and

$$f(m+1) = \left[\frac{m^2 + 2m + 1}{2m+1}\right] = 1 + \left[\frac{m^2}{2m+1}\right], \; f(1) = \left[\frac{1}{2m+1}\right] = 0.$$

By the same way as above, using inequality $f(k+1) \leq f(k) + 1$, can be proved that for any $0 < i < 1 + \left[\frac{m^2}{2m+1}\right]$ there is preimage in I_{m+1}.
So,

$$|f(I_{m+1})| = 1 + \left[\frac{m^2}{2m+1}\right].$$

Remains consider behavior of f on $I_{2m+1} \setminus I_{m+1} = \{m+1+k \mid k \in I_m\}$.
For any $k \in I_m$ we have

$$f(m+1+k) = \left[\frac{(m+1)^2 + 2(m+1)k + k^2}{2m+1}\right] =$$

$$= k + \left[\frac{k^2 + k + (m+1)^2}{2m+1}\right]$$

and then

$$f(m+1+(k+1)) \geq k+1+\left[\frac{(k+1)^2 + (k+1) + (m+1)^2}{2m+1}\right] >$$

$$> k + \left[\frac{k^2 + k + (m+1)^2}{2m+1}\right] = f(m+1+k).$$

Since $f(k)$ is strictly increasing in $k \in I_{2m+1} \setminus I_{m+1}$ then

$$|f(I_{2m+1} \setminus I_{m+1})| = m.$$

Thus,

$$|f(I_{2m+1})| = m + 1 + \left[\frac{m^2}{2m+1}\right] = 1 + \left[\frac{3m^2+m}{2m+1}\right].$$

So,

$$|f(I_n)| = \begin{cases} 1 + \left[\frac{3m}{2}\right] & \text{if } n = 2m \\ 1 + \left[\frac{3m^2+3m+1}{2m+1}\right] & \text{if } n = 2m+1 \end{cases}.$$

For $n = 2m$ we have

$$|f(I_n)| = 1 + \left[\frac{3m}{2}\right] = 1 + \left[\frac{6m}{4}\right] = 1 + \left[\frac{3n}{4}\right].$$

For $n = 2m+1$ we have

$$|f(I_n)| = 1 + \left[\frac{3m^2+3m+1}{2m+1}\right] = 1 + \left[\frac{\frac{12m^2+12m+4}{2m+1}}{4}\right] =$$

$$= 1 + \left[\frac{\frac{3(2m+1)^2+1}{2m+1}}{4}\right] = 1 + \left[\frac{\left[3(2m+1) + \frac{1}{2m+1}\right]}{4}\right] =$$

$$= 1 + \left[\frac{3(2m+1)}{4}\right] = 1 + \left[\frac{3n}{4}\right].$$

So,

$$|f(I_n)| = 1 + \left[\frac{3n}{4}\right].$$

Arkady Alt

W17. (Solution by the proposer.) First we solve the recurrence

$$t_{k+2} - t_{k+1} - t_k = f_k, k \in \mathbb{N}, \text{where} f_k, k \in \mathbb{N} \text{ are Fibonacci numbers.} \quad (1)$$

For any sequence $(t_n)_{n \geq 0}$ we set $F(t_n) := t_{n+2} - t_{n+1} - t_n$ and call F Fibonacci Operator.
Easy to see that F is linear operator, that is
$F(at'_n + bt''_n) = aF(t'_n) + bF(t''_n)$ for any $a, b \in \mathbb{R}$.
Noting that

$$F(nf_n) = (n+2)f_{n+2} - (n+1)f_{n+1} - nf_n =$$

$$= 2f_{n+2} - f_{n+1} = f_{n+1} + 2f_n$$

and

$$F(nf_{n-1}) = (n+2)f_{n+1} - (n+1)f_n - nf_{n-1} =$$