Problem W16.
(**J.Wildt IMC**, **OCTOGON Mathematical Magazine vol.25**,**n.1**,**2017**, **p.242**).
Problem with a solution proposed by Arkady Alt, **San Jose**, **California**, **USA**
Find number of elements in image of function

$$k \mapsto \left[\frac{k^2}{n}\right] : \{1,2,\ldots,n\} \to \mathbb{N} \cup \{0\}.$$

Solution.
Let $I_n := \{1,2,\ldots,n\}$ and $f(k) := \left[\frac{k^2}{n}\right]$, for any $k \in I_n$. We have to dtermine $|f(I_n)|$.
Consider two cases.
Case1. n is even, that is $n = 2m$.
Lemma.
For any $k \in I_m$ holds inequality $f(k+1) - f(k) \le 1$.
Proof.
First we consider $k \in I_{m-1}$. Let $1 \le k \le m-1$ then $f(k+1) = \left[\frac{k^2+2k+1}{2m}\right]$ and $1 + f(k) = \left[\frac{k^2+2m}{2m}\right]$. Note that $k \le m-1$ yields

$$f(k+1) \le \left[\frac{k^2+2(m-1)+1}{2m}\right] = \left[\frac{k^2+2m-1}{2m}\right] \le \left[\frac{k^2+2m}{2m}\right] = 1 + f(k).$$

Also for $k = m$ we have $f(m+1) = \left[\frac{m^2+2m+1}{2m}\right] = \left[\frac{\frac{m^2+2m+1}{2}}{m}\right] =$

$$\left[\frac{\left[\frac{m^2+2m+1}{m}\right]}{2}\right] = \left[\frac{m+2+\left[\frac{1}{m}\right]}{2}\right] = 1 + \left[\frac{m}{2}\right] = 1 + f(m). \blacksquare$$

Corollary.

$$f(I_m) = \left\{0,1,2,\ldots,\left[\frac{m}{2}\right]\right\}.$$

Proof.
Note that $f(k)$ isn't decreasing, that is $f(k+1) - f(k) = \left[\frac{(k+1)^2}{n}\right] - \left[\frac{k^2}{n}\right] \ge 0.$
Also, $f(m) = \left[\frac{m^2}{2m}\right] = \left[\frac{m}{2}\right]$ and $f(1) = \left[\frac{1}{2m}\right] = 0.$
Suppose that there is $i \in I_{m/2}$ for which $f^{-1}(i) = \varnothing$. Obvious that $1 \le i < \left[\frac{m}{2}\right]$.
Let $k_* := \{k \mid k \in I_m \text{ and } f(k) < i\}$.
Then $f(k_*) < i < f(k_*+1) \Rightarrow f(k_*+1) - f(k_*) > 1$, that is contradiction to **Lemma.** $\blacksquare$
Now note that $f(k)$ is strictly increasing in $k \in \{m+1, m+2, .., 2m\}$.
Indeed, since for any $k \in I_m$ we have $f(m+k) = \left[\frac{(m+k)^2}{2m}\right] = \left[\frac{m^2+2mk+k^2}{2m}\right] =$
$k + \left[\frac{m^2+k^2}{2m}\right]$ then $f(m+(k+1)) = k+1+\left[\frac{m^2+(k+1)^2}{2m}\right] >$

$$k + \left[\frac{m^2+k^2}{2m}\right] = f(m+k) \text{ for any } k \in I_{m-1}.$$

Hence, $|f(I_{2m} \setminus I_m)| = m$ and since $|f(I_m)| = \left[\frac{m}{2}\right] + 1$ then

$|f(I_{2m})| = m + \left[\frac{m}{2} \right] + 1 = \left[\frac{3m+2}{2} \right].$

Case 2. n is odd, that is $n = 2m + 1$.

Then as above we will prove divide this case on two parts.

First we consider f on I_{m+1}.

For any $k \in I_m$ we have

$f(k+1) = \left[\frac{k^2+2k+1}{2m+1} \right] \le \left[\frac{k^2+2m+1}{2m+1} \right] = 1 + \left[\frac{k^2}{2m+1} \right] = 1 + f(k+1)$

and $f(m+1) = \left[\frac{m^2+2m+1}{2m+1} \right] = 1 + \left[\frac{m^2}{2m+1} \right], f(1) = \left[\frac{1}{2m+1} \right] = 0.$

By the same way as above, using inequality $f(k+1) \le f(k) + 1$, can be proved that for any $0 < i < 1 + \left[\frac{m^2}{2m+1} \right]$ there is preimage in I_{m+1}.

So, $|f(I_{m+1})| = 1 + \left[\frac{m^2}{2m+1} \right].$

Remains consider behavior of f on $I_{2m+1} \setminus I_{m+1} = \{m+1+k \mid k \in I_m\}$.

For any $k \in I_m$ we have $f(m+1+k) = \left[\frac{(m+1)^2 + 2(m+1)k + k^2}{2m+1} \right] =$

$k + \left[\frac{k^2 + k + (m+1)^2}{2m+1} \right]$ and then

$f(m+1+(k+1)) \ge k + 1 + \left[\frac{(k+1)^2 + (k+1) + (m+1)^2}{2m+1} \right] > k + \left[\frac{k^2 + k + (m+1)^2}{2m+1} \right] =$

$f(m+1+k)$. Since $f(k)$ is strictly increasing in $k \in I_{2m+1} \setminus I_{m+1}$ then

$|f(I_{2m+1} \setminus I_{m+1})| = m$. Thus, $|f(I_{2m+1})| = m + 1 + \left[\frac{m^2}{2m+1} \right] = 1 + \left[\frac{3m^2+m}{2m+1} \right].$

So, $|f(I_n)| = \begin{cases} 1 + \left[\frac{3m}{2} \right] & \text{if } n = 2m \\ 1 + \left[\frac{3m^2+3m+1}{2m+1} \right] & \text{if } n = 2m+1 \end{cases}.$

For $n = 2m$ we have $|f(I_n)| = 1 + \left[\frac{3m}{2} \right] = 1 + \left[\frac{6m}{4} \right] = 1 + \left[\frac{3n}{4} \right].$

For $n = 2m+1$ we have $|f(I_n)| = 1 + \left[\frac{3m^2+3m+1}{2m+1} \right] = 1 + \left[\frac{\frac{12m^2+12m+4}{2m+1}}{4} \right] =$

$1 + \left[\frac{\frac{3(2m+1)^2+1}{2m+1}}{4} \right] = 1 + \left[\frac{\left[3(2m+1) + \frac{1}{2m+1} \right]}{4} \right] =$

$1 + \left[\frac{3(2m+1)}{4} \right] = 1 + \left[\frac{3n}{4} \right].$

So, $|f(I_n)| = 1 + \left[\frac{3n}{4} \right].$