W15. (Solution by the proposer.) Let

$$\sigma_{m,k}(n) := \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_k=1}^{n}\overset{m}{\min}\{i_1,i_2,...,i_k\}.$$

Since

$$\overset{m}{\min}\{i_1,i_2,...,i_k\} = \sum_{t=1}^{\min^m\{i_1,i_2,...,i_k\}} 1$$

and

$$1\le t\le \overset{m}{\min}\{i_1,i_2,...,i_k\} \iff \begin{cases} 1\le t\le i_1^m \\ 1\le t\le i_2^m \\ .... \\ 1\le t\le i_k^m \end{cases}$$

then

$$\sigma_{m,k}(n) = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_k=1}^{n}\sum_{t=1}^{\min^m\{i_1,i_2,...,i_k\}} 1 = |D_{ext}|,$$

where mD_{ext} is set of all $k+1$-tipples $(t,i_1,i_2,...,i_k)$ which satisfy to system of inequalities

$$\begin{cases} 1\le i_r\le n, r=1,2,...,k \\ 1\le t\le i_r^m, r=1,2,...,k \end{cases} \iff \begin{cases} 1\le t\le n^m \\ \lceil \sqrt[m]{t}\rceil \le i_r\le n, r=1,2,...,k \end{cases}.$$

Hence,

$$\sigma_{m,k}(n) = \sum_{i=1}^{n^m}\sum_{i_1=\lceil \sqrt[m]{t}\rceil}^{n}\sum_{i_2=\lceil \sqrt[m]{t}\rceil}^{n}\cdots\sum_{i_k=\lceil \sqrt[m]{t}\rceil}^{n} = \sum_{i=1}^{n^m}\left(n-\lceil \sqrt[m]{t}\rceil+1\right)^k.$$

Since

$$\{1,2,...,n^m\} = \bigcup_{p=0}^{n-1}\{p^m+1,(p+1)^m\}$$

and $\lceil \sqrt[m]{t}\rceil = p+1$ for $t\in\{p^m+1,(p+1)^m\}$ then

$$\sigma_{m,k}(n)=\sum_{p=0}^{n-1}\sum_{t=p^m+1}^{(p+1)^m}\left(n-\left\lceil\sqrt[m]{t}\right\rceil+1\right)^k=\sum_{p=0}^{n-1}\sum_{t=p^m+1}^{(p+1)^m}(n-p)^k=$$

$$=\sum_{p=0}^{n-1}\left((p+1)^m-p^m\right)(n-p)^k.$$

In particular,

$$\sigma_{1,k}(n)=\sum_{p=0}^{n-1}\left((p+1)-p\right)(n-p)^k=\sum_{p=0}^{n-1}(n-p)^k=\sum_{p=1}^{n}p^k,$$

$$\sigma_{2,k}(n)=\sum_{p=0}^{n-1}\left((p+1)^2-p^2\right)(n-p)^k=\sum_{p=0}^{n-1}(2p+1)(n-p)^k=$$

$$=\sum_{p=1}^{n}\left(2(n-p)+1\right)p^k=(2n+1)\sum_{p=1}^{n}p^k-2\sum_{p=1}^{n}p^{k+1}.$$

Note that

$$\sigma_{m,k}(n)=\sum_{p=0}^{n-1}\left((p+1)^m-p^m\right)(n-p)^k=$$

$$=\sum_{p=0}^{n-1}(p+1)^m\left(n+1-(p+1)\right)^k-\sum_{p=0}^{n-1}p^m(n-p)^k=$$

$$=\sum_{p=1}^{n}p^m(n+1-p)^k-\sum_{p=1}^{n-1}p^m(n-p)^k.$$

Let $\delta_{m,k}(n):=\sum_{p=1}^{n-1}p^m(n-p)^k$ then $\sigma_{m;k}(n)=\delta_{m,k}(n+1)-\delta_{m,k}(n)$.
Also note that

$$\delta_{m,k}(n)=\sum_{p=1}^{n-1}p^m(n-p)^k=\sum_{p=1}^{n-1}\left(n-(n-p)\right)^m(n-p)^k=$$

$$=\sum_{q=1}^{n-1}(n-q)^m q^k=\delta_{k,m}(n)$$

.

and

$$\delta_{m,k+1}(n) + \delta_{m+1,k}(n) = \sum_{p=1}^{n-1} p^m (n-p)^{k+1} + \sum_{p=1}^{n-1} p^{m+1} (n-p)^k =$$

$$= \sum_{p=1}^{n-1} p^m (n-p)^k (n-p+p) = n\delta_{m,k}(n).$$

In particular,

$$\delta_{1,k}(n) = \sum_{p=1}^{n-1} p (n-p)^k = \sum_{p=1}^{n-1} (n-p) p^k =$$

$$= n\sum_{p=1}^{n-1} p^k - \sum_{p=1}^{n-1} p^{k+1} = ns_k(n-1) - s_{k+1}(n-1)$$

checking

$$\sigma_{1,k}(n) =$$

$$= \delta_{1,k}(n+1) - \delta_{1,k}(n) = (n+1) s_k(n) - s_{k+1}(n) - ns_k(n-1) + s_{k+1}(n-1) =$$

$$= s_k(n) + n(s_k(n) - s_k(n-1)) - (s_{k+1}(n) - s_{k+1}(n-1)) =$$

$$= s_k(n) + n \cdot n^k - n^{k+1} = s_k(n)$$

and

$$\delta_{2,k}(n) = n\delta_{1,k}(n) - \delta_{1,k+1}(n) =$$

$$= n(ns_k(n-1) - s_{k+1}(n-1)) - ns_{k+1}(n-1) + s_{k+2}(n-1) =$$

$$= n^2 s_k(n-1) - 2ns_{k+1}(n-1) + s_{k+2}(n-1).$$

Suppose that

$$\delta_{m,k}(n) = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} n^i s_{k+m-i}(n-1)$$

.

then

$$\delta_{m+1,k}(n) = n\delta_{m,k}(n) - \delta_{m,k+1}(n) =$$

$$= \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} n^{i+1} s_{k+m-i}(n-1) - \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} n^{i} s_{k+1+m-i}(n-1) =$$

$$= \sum_{i=0}^{m} (-1)^{m+1-(i+1)} \binom{m}{i} n^{i+1} s_{k+m+1-(i+1)}(n-1) -$$

$$- \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} n^{i} s_{k+1+m-i}(n-1) =$$

$$= \sum_{i=1}^{m+1} (-1)^{m+1-i} \binom{m}{i-1} n^{i} s_{k+m+1-i}(n-1) +$$

$$+ \sum_{i=1}^{m} (-1)^{m+1-i} \binom{m}{i} n^{i} s_{k+1+m-i}(n-1) + (-1)^{m+1} s_{k+1+m}(n-1) =$$

$$= \sum_{i=1}^{m+1} (-1)^{m+1-i} n^{i} s_{k+m+1-i}(n-1) \left(\binom{m}{i-1} + \binom{m}{i} \right) +$$

$$+ (-1)^{m+1} s_{k+1+m}(n-1) =$$

$$= \sum_{i=1}^{m+1} (-1)^{m+1-i} n^{i} s_{k+m+1-i}(n-1) \binom{m+1}{i} + (-1)^{m+1} s_{k+m+1}(n-1) =$$

$$= \sum_{i=0}^{m+1} (-1)^{m+1-i} \binom{m+1}{i} n^{i} s_{k+m+1-i}(n-1) .$$

Or, using formal operators, we can obtain the same result much shorter: Since $\delta_{m+1,k}(n) = n\delta_{m,k}(n) - \delta_{m,k+1}(n)$ and $\delta_{1,k}(n) = ns_k(n-1) - s_{k+1}(n-1)$ then using identical operator I defined by $I(a_k) = a_k$ and shift operator S defined by $S(a_k) = a_{k+1}$ we obtain $\delta_{1,k}(n) = (n \cdot I - S)(s_k(n-1))$, $\delta_{2,k}(n) = (n \cdot I - S)^2 (s_k(n-1))$. Since from supposition $\delta_{m,k}(n) = (nI - S)^m (s_k(n-1))$ follow

$$\delta_{m+1,k}(n) = (nI - S)(\delta_{m,k}(n)) = (nI - S)(nI - S)^m (s_k(n-1)) =$$

$$= (nI - S)^{m+1} (s_k(n-1)).$$

Hence

$$\sigma_{m,k}(n) = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \left((n+1)^i s_{k+m-i}(n) - n^i s_{k+m-i}(n-1)\right).$$

Since $s_{k+m-i}(n-1) = s_{k+m-i}(n) - n^{k+m-i}$ then

$$\sigma_{m,k}(n) =$$

$$= \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \left((n+1)^i s_{k+m-i}(n) - n^i \left(s_{k+m-i}(n) - n^{k+m-i}\right)\right) =$$

$$= \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \left(\left((n+1)^i - n^i\right) s_{k+m-i}(n) + n^i \cdot n^{k+m-i}\right) =$$

$$= \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \left((n+1)^i - n^i\right) s_{k+m-i}(n) +$$

$$+ n^{k+m} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \left((n+1)^i - n^i\right) s_{k+m-i}(n) =$$

$$= \sum_{i=1}^{m} (-1)^{m-i} \binom{m}{i} \left((n+1)^i - n^i\right) s_{k+m-i}(n).$$

Thus,finally

$$\sigma_{m,k}(n) = \sum_{i=1}^{m} (-1)^{m-i} \binom{m}{i} \left((n+1)^i - n^i\right) s_{k+m-i}(n) =$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{i} (-1)^{m-i} \binom{m}{i} \binom{i}{j} n^{i-j} s_{k+m-i}(n).$$

As example we apply obtained formula to determine $\sigma_{3,3}(n)$:

$$\sigma_{3,3}(n)=\sum_{i=1}^{3}(-1)^{3-i}\binom{3}{i}\left((n+1)^i-n^i\right)s_{k+m-i}(n)=\left((n+1)^3-n^3\right)s_3(n)-$$

$$-3\left((n+1)^2-n^2\right)s_4(n)+3\left((n+1)-n\right)s_5(n)=$$

$$=\left(3n^2+3n+1\right)s_3(n)-3(2n+1)s_4(n)+3s_5(n).$$

Since

$$s_4(n)=\frac{n(n+1)(2n+1)\left(3n^2+3n-1\right)}{30},s_5(n)=\frac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12}$$

then

$$\sigma_{3,3}(n)=\frac{n^2(n+1)^2\left(3n^2+3n+1\right)}{4}-\frac{n(n+1)(2n+1)^2\left(3n^2+3n-1\right)}{10}+$$

$$+\frac{n^2(n+1)^2\left(2n^2+2n-1\right)}{4}=\frac{1}{20}n(n+1)\left(n^2+1\right)\left(n^2+2n+2\right).$$

Arkady Alt

W16. (Solution by the proposer.) Let $I_n := \{1,2,...,n\}$ and $f(k) := \left[\frac{k^2}{n}\right]$,for any $k \in I_n$. We have to dtermine $|f(I_n)|$.
Consider two cases.

Case1. n is even, that is $n = 2m$.
Lemma. For any $k \in I_m$ holds inequality

$$f(k+1)-f(k)\leq 1.$$

Proof. First we consider $k \in I_{m-1}$. Let $1 \leq k \leq m-1$then

$$f(k+1)=\left[\frac{k^2+2k+1}{2m}\right]$$