standard theory (see here for example
https://artofproblemsolving.com/community/c13188h278791_the_uvwmethod)
states that the minimum of W^3 and then of W occurs when $X = Y$ or $Y = Z$ or $Z = X$ and this in turn means that $a = b$ or $b = c$ or $c = a$.
Coming back to

$$\Delta(a,b,c)\Delta(a^3,b^3,c^3) - \Delta(a^4,b^4,c^4) \geq 0,$$

if we set $a = b$, we get

$$4c^4(b^2 + bc + c^2)(-c + b)^2 \geq 0$$

and this clearly holds true.

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W14. (Solution by the proposer.) Let

$$\widetilde{D}_1 = \{x_N \mid x_N \in D_1 \text{ and } S(x_N) \neq \infty\}.$$

Since $\widetilde{D}_1$ isn't empty (because for for instance if $x_n = q^{n-1}, n \in N$, where $q \in (0,1)$ we have

$$\sum_{n=1}^{\infty} f(x_n, x_{n+1}) = \sum_{n=1}^{\infty} f(q^{n-1}, q^n) =$$

$$= f(1,q)\sum_{n=1}^{\infty} q^{m(n-1)} = \frac{f(1,q)}{1-q^m}$$

then

$$\inf\{S(x_N) \mid x_N \in D_1\} = \inf\left\{S(x_N) \mid x_N \in \widetilde{D}_1\right\}.$$

Let $S := \inf\left\{S(x_N) \mid x_N \in \widetilde{D}_1\right\}$. For any $x_N \in \widetilde{D}_1$ we have

$$S(x_N) = \sum_{n=1}^{\infty} f(x_n, x_{n+1}) =$$

$$= f(1,x_2) + \sum_{n=2}^{\infty} f(x_n, x_{n+1}) = f(1,x_2) + x_2^m \sum_{n=1}^{\infty} f(y_n, y_{n+1}) =$$

$$= f(1,x_2) + x_2^m S_f(y_{\mathbb{N}}),$$

where $y_n := \dfrac{x_{n+1}}{x_2}, n \in N$.
Since $y_N \in \widetilde{D}_1 (1 = y_1 > y_2 > .... > y_n > ...$ and
$S(y_N) = \dfrac{S_f(x_N)}{x_2^m} - f(1, x_2))$ then $S_f(y_N) \geq S$ and, therefore,

$$S(x_N) \geq f(1, x_2) + x_2^m S \implies S \geq f(1, x_2) + x_2^m S \iff$$

$$S \geq \frac{f(1, x_2)}{1 - x_2^m} \geq \mu,$$

where

$$\mu := \min_{x \in [0,1)} \frac{f(1, x)}{1 - x^m} = \frac{f(1, x_*)}{1 - x_*^m}, x_* \in [0, 1).$$

Since in the case $0 < x_* < 1$ for $x_n = x_*^{n-1}, n \in N$ we have

$$S_f(x_N) = \frac{f(1, x_*)}{1 - x_*^m} = \mu$$

then μ is attainable in $\widetilde{D}_1$ lover bound for $S_f(x_N)$,that is

$$\inf \{S_f(x_N) \mid x_N \in D_1\} = \min \{S_f(x_N) \mid x_N \in D_1\} = \mu.$$

In the case $x_* = 0$ lower bound $\mu = f(1, 0)$ isn't attainable because
$x_2 > 0 \implies S \geq \dfrac{f(1, x_2)}{1 - x_2^m} > \mu$.
But, for any $\varepsilon > 0$ we can find q such that $S_f(x_N) < \mu + \varepsilon$ if
$x_n = q^{n-1}, n = 1, 2, ...$
Indeed, since

$$\lim_{q \to 0+} \frac{f(1, q)}{1 - q^m} = f(1, 0) < \frac{f(1, q)}{1 - q^m}$$

then there is $\delta > 0$ such that for any $q \in (0, \delta)$ holds

$$\frac{f(1, q)}{1 - q^m} - f(1, 0) < \varepsilon \iff \frac{f(1, q)}{1 - q^m} < f(1, 0) + \varepsilon.$$

It is mean that obtained lower bound $f(1, 0)$ for $S_f(x_N)$ is exact lower bound, that is infimum.

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