$$T_{2n+1} := \frac{u^{2n+1} - v^{2n+1}}{u - v} + \frac{v^{2n+1} - w^{2n+1}}{v - w} + \frac{w^{2n+1} - u^{2n+1}}{w - u}$$

We have successively,

$$T_{2n+1} = \left(u^{2n} + u^{2n-1}v + uv^{2n-1} + v^{2n}\right) + \left(v^{2n} + v^{2n-1}w + vw^{2n-1} + w^{2n}\right) +$$

$$+ \left(w^{2n} + w^{2n-1}u + wu^{2n-1} + u^{2n}\right) =$$

$$= 2S^{(2n)} + \sum_{k=1}^{2n} \left(u^{2n-k}v^k + v^{2n-k}w^k + w^{2n-k}u^k\right) =$$

$$= 2S^{(2n)} + \sum_{k=1}^{n} (u^{2n-k}v^k + u^k v^{2n-k} + v^{2n-k}w^k + v^k w^{2n-k} +$$

$$+ w^{2n-k}u^k + w^k u^{2n-k}) = 2S^{(2n)} + \sum_{k=1}^{n} [(u^{2n-k} + v^{2n-k} + w^{2n-k}) \cdot$$

$$\cdot \left(u^k + v^k + w^k\right) - \left(u^{2n} + v^{2n} + w^{2n}\right)] =$$

$$2S^{(2n)} + \sum_{k=1}^{n} \left(S^{(2n-k)}S^{(k)} - S^{(2n)}\right) = (2 - n)\, S^{(2n)} + \sum_{k=1}^{n} S^{(2n-k)}S^{(k)}$$

How the sums $S^{(k)}$ are integers, then the sum T_{2n+1} is also an integer.

W28. Solution by the proposer. Since

$$F_{n+1} \cdot F_{n-1} - F_n^2 = (-1)^n \text{ (Cassini identity)}$$

then for

$$(x_0, y_0) = ((-1)^n F_{n-1}, (-1)^n F_n)$$

we have

$$F_{n+1}x_0 - F_n y_0 = F_{n+1} \cdot (-1)^n F_{n-1} - F_n \cdot (-1)^n F_n =$$

$$= (-1)^n \left(F_{n+1} \cdot F_{n-1} - F_n^2\right) = 1$$

and, therefore

$$F_{n+1}x - F_n y = 1 \Leftrightarrow F_{n+1}x - F_n y = F_{n+1} \cdot (-1)^n F_{n-1} - F_n \cdot (-1)^n F_n \Leftrightarrow$$

$$\Leftrightarrow F_{n+1}\left(x - (-1)^n F_{n-1}\right) = F_n\left(y - (-1)^n F_n\right) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x - (-1)^n F_{n-1} = tF_n \\ y - (-1)^n F_{n-1} = tF_{n+1} \end{cases} \Leftrightarrow \begin{cases} x = tF_n + (-1)^n F_{n-1} \\ y = tF_{n+1} + (-1)^n F_n \end{cases}$$

$t \in Z$. Thus

$$D_n = \{(x, y)\,|x = tF_n + (-1)^n F_{n-1}, y = tF_{n+1} + (-1)^n F_n, t \in Z\}$$

a).

$$\min_{(x,y)\in D_n} |x + y| = \min_{t\in Z} |t(F_n + F_{n+1}) + (-1)^n (F_{n-1} + F_n)| = \min_{t\in Z} \varphi(t)$$

where $\varphi(t) := |tF_{n+2} + (-1)^n F_{n+1}|$.
Since

$$tF_{n+2} + (-1)^n F_{n+1} = 0 \Leftrightarrow t = t_* := \frac{(-1)\,F_{n+1}}{F_{n+2}}$$

and

$$|t_*| = \frac{F_{n+1}}{F_{n+2}} < 1$$

then integer t which minimize $\varphi(t)$ must be among to t_* integers, that is.
Thus

$$\min_{t\in Z} \varphi(t) = \min_{t\in\{-1,0,1\}} \varphi(t) = \min\{\varphi(0), \varphi(1), \varphi(-1)\} =$$

$$= \min\{F_{n+1}, |F_{n+2} + (-1)^n F_{n+1}|, |-F_{n+2} + (-1)^n F_{n+1}|\}$$

If n is odd then

$$\min\{F_{n+1}, |F_{n+2} + (-1)^n F_{n+1}|, |-F_{n+2} + (-1)^n F_{n+1}|\} =$$

$$= \min\{F_{n+1}, F_{n+2} - F_{n+1}, F_{n+2} + F_{n+1}\} =$$

$$= \min\left\{F_{n+1}, F_n, F_{n+3}\right\} = \varphi(1)$$

If n is even then

$$\min\left\{F_{n+1}, F_{n+2} + F_{n+1}, \left|-F_{n+2} + F_{n+1}\right|\right\} =$$

$$= \min\left\{F_{n+1}, F_{n+3}, F_n\right\} = \varphi(-1)$$

So,

$$\min_{(x,y)\in D_n} |x+y| = F_n$$

b).

$$|x| + |y| = \left|tF_n + (-1)^n F_{n-1}\right| + \left|tF_{n+1} + (-1)^n F_n\right|$$

By replacing t with $(-1)^{n-1} t$ we obtain

$$|x| + |y| = \left|tF_n - F_{n-1}\right| + \left|tF_{n+1} - F_n\right| =$$

$$\max\left\{\left|tF_n - F_{n-1} + tF_{n+1} - F_n\right|, \left|tF_n - F_{n-1} - tF_{n+1} + F_n\right|\right\} =$$

$$= \max\left\{\left|t\left(F_n + F_{n+1}\right) - \left(F_{n-1} + F_n\right)\right|, \left|t\left(F_n - F_{n+1}\right) + F_n - F_{n-1}\right|\right\} =$$

$$= \max\left\{\left|tF_{n+2} - F_{n+1}\right|, \left|-F_{n-1}t + F_{n-2}\right|\right\} =$$

$$= \max\left\{\left|tF_{n+2} - F_{n+1}\right|, \left|F_{n-1}t - F_{n-2}\right|\right\}$$

Lemma. Let $p, q > 0$, $a, b \in R$. Then

$$\min_{x\in R}\left(\max\left\{|px - a|, |qx - b|\right\}\right) = \frac{|aq - bp|}{p+q}$$

and attained if $x = \frac{a+b}{p+q}$.

Proof. First note $\min\limits_{x\in R}\left(\max\left\{|px - a|, |qx - b|\right\}\right) = \min t$ where t provide solvability of inequality

$$t \geq \max\{|px-a|, |qx-b|\} \Leftrightarrow \begin{cases} t \geq |px-a| \\ t \geq |qx-b| \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} -t \leq px-a \leq t \\ -t \leq qx-b \leq t \end{cases} \Leftrightarrow \begin{cases} \frac{a-t}{p} \leq x \leq \frac{t+a}{p} \\ \frac{b-t}{q} \leq x \leq \frac{t+b}{q} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \max\left\{\frac{a-t}{p}, \frac{b-t}{q}\right\} \leq x \leq \min\left\{\frac{t+a}{p}, \frac{t+b}{q}\right\} \tag{1}$$

Condition of solvability is

$$\begin{cases} \frac{a-t}{p} \leq \frac{t+b}{q} \\ \frac{b-t}{q} \leq \frac{t+a}{p} \end{cases} \Leftrightarrow \begin{cases} q(a-t) \leq p(t+b) \\ p(b-t) \leq q(t+a) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \frac{aq-pb}{p+q} \leq t \\ \frac{pb-aq}{p+q} \leq t \end{cases} \Leftrightarrow \frac{|aq-bp|}{p+q} \leq t$$

Thus

$$\min_{x \in R}(\max\{|px-a|, |qx-b|\}) = \frac{|aq-bp|}{p+q}$$

Assuming WLOG that $aq \geq bp$, by substitution $t = \frac{aq-pb}{p+q}$ in inequality (1) we obtain $x = \frac{a+b}{p+q}$. Indeed,

$$\frac{t+a}{p} = \frac{\frac{aq-pb}{p+q}+a}{p} = \frac{2aq+ap-bp}{p(p+q)}, \frac{t+b}{q} = \frac{\frac{aq-pb}{p+q}+b}{q} = \frac{a+b}{p+q}$$

and

$$\frac{2aq+ap-bp}{p(p+q)} - \frac{a+b}{p+q} = \frac{2}{p} \cdot \frac{aq-pb}{p+q} \geq 0$$

Hence $x \leq \frac{a+b}{p+q}$. Similarly we can obtain $\frac{a+b}{p+q} \leq x$. So, $x = \frac{a+b}{p+q}$.

Corollary. Let $p > q$ then

$$\min_{x \in R}(|px-a| + |qx-b|) = \frac{|aq-bp|}{p+q}$$

Proof. Since

$$|px-a|+|qx-b| = \max\{|(p+q)x-(a+b)|, |(p-q)x-(a-b)|\}$$

then by Lemma

$$|px-a|+|qx-b| \geq \frac{|(a+b)(p-q)-(a-b)(p+q)|}{(p+q)+(p-q)} = \frac{|aq-bp|}{p}$$

and equality occurs iff

$$x = \frac{(a+b)+(a-b)}{(p+q)+(p-q)} = \frac{2a}{2p} = \frac{a}{p}$$

So,

$$\min_{x\in R}(|px-a|+|qx-b|) = \frac{|aq-bp|}{p}$$

and attained if $x=\frac{a}{p}$.

Application. Since

$$|x|+|y| = |tF_n - F_{n+1}| + |tF_{n+1} - F_n|$$

and $F_{n+1} > F_n$ then by Corollary for real t minimum of $|tF_n - F_{n+1}| + |tF_{n+1} - F_n|$ attained if $t_* = \frac{F_n}{F_{n+1}} < 1$ and closest to t_* integer values of t are $t=1$ and $t=0$. Therefore,

$$\min_{t\in Z}|tF_n - F_{n+1}| + |tF_{n+1} - F_n| =$$

$$= \min\{|0\cdot F_n - F_{n+1}| + |0\cdot F_{n+1} - F_n|, |1\cdot F_n - F_{n+1}| + |1\cdot F_{n+1} - F_n|\} =$$

$$= \min\{F_{n-1}+F_n, F_n - F_{n-1} + F_{n+1} - F_n\} = \min\{F_{n+1}, F_n\} = F_n$$

c).

$$x^2+y^2 = (tF_n - F_{n-1})^2 + (tF_{n+1} - F_n)^2 =$$

$$= \left(F_n^2 + F_{n+1}^2\right)t^2 - 2F_n(F_{n-1}+F_{n+1})t + \left(F_n^2+F_{n-1}^2\right)$$

Since

$$F_{n-1}^2 + F_n^2 = F_{2n-1} \text{ and } F_n\left(F_{n+1} + F_{n-1}\right) = F_{2n}$$

then

$$x^2 + y^2 = F_{2n+1}t^2 - 2F_{2n}t + F_{2n-1}$$

Quadratic trinomial $F_{2n+1}t^2 - 2F_{2n}t + F_{2n-1}$ attain minimum in R when $t = \frac{F_{2n}}{F_{2n+1}} < 1$ and, therefore, minimum in Z can be attained in one of two integer points closest to $\frac{F_{2n}}{F_{2n+1}}$, that is when $t = 0$ or $t = 1$.
Thus

$$\min\left(x^2 + y^2\right) =$$

$$= \min\left\{F_{2n+1} \cdot 0^2 - 2F_{2n} \cdot 0 + F_{2n-1}, F_{2n+1} \cdot 1^2 - 2F_{2n} \cdot 1 + F_{2n-1}\right\} =$$

$$= \min\left\{F_{2n-1}, F_{2n+1} - 2F_{2n} + F_{2n-1}\right\} = \min\left\{F_{2n-1}, F_{2n-3}\right\} = F_{2n-3}$$

W29. Solution by the proposer. Let P be a point in $\triangle ABC$ with barycentric coordinates

$$(p_a, p_b, p_c) = (x, y, z)$$

Let $R_a := PA$, $R_b = PB$, $R_c := PC$ and A_p, B_p, C_p be foots of perpendiculars from P to sides BC, CA, AB respectively. Also, we denote via

$$d_a := PA_p, d_b := PB_p, d_c := PC_p$$

and

$$a_p := B_pC_p, b_p := C_pA_p, c_p := A_pB_p$$

(side lengths of pedal triangle $A_pB_pC_p$). Let $F := [ABC]$ and

$$F_a := [PBC], F_b := [PCA], F_c := [PAB].$$

Then

$$F_a = \frac{ad_a}{2}, F_b = \frac{bd_b}{2}, F_c = \frac{cd_c}{2}$$

and

$$p_a = \frac{F_a}{F} = \frac{ad_a}{2F}, p_b = \frac{F_b}{F} = \frac{bd_b}{2F}, p_c = \frac{F_c}{F} = \frac{cd_c}{2F}$$

Since $\angle B_pPC_p = 180^\circ - A$ then, by Cos-Theorem

$$a_p^2 = d_b^2 + d_c^2 + 2d_bd_c \cos A$$

and $\cos A = \frac{b^2+c^2-a^2}{2bc}$.
Using $d_b = \frac{2p_bF}{b}, d_c = \frac{2p_cF}{c}$ and $\cos A = \frac{b^2+c^2-a^2}{2bc}$ we obtain:

$$a_p^2 = \frac{4p_b^2F^2}{b^2} + \frac{4p_c^2F^2}{c^2} + \frac{4p_bp_cF^2}{bc} \cdot \frac{b^2+c^2-a^2}{bc} =$$

$$= \frac{4F^2}{b^2c^2}\left(p_b^2c^2 + p_c^2b^2 + p_bp_c\left(b^2+c^2-a^2\right)\right) =$$

$$= \frac{4F^2}{b^2c^2}\left(p_b\left(1-p_c-p_a\right)c^2 + p_c\left(1-p_a-p_b\right)b^2 + p_bp_c\left(b^2+c^2-a^2\right)\right) =$$

$$= \frac{4F^2}{b^2c^2}\left(p_bc^2 + p_cb^2 - p_bp_ca^2 - p_cp_ab^2 - p_ap_bc^2\right)$$

Since $abc = 4FR$ then

$$\frac{4F^2}{b^2c^2} = \frac{4a^2F^2}{a^2b^2c^2} = \frac{a^2F^2}{4R^2}$$

and, therefore

$$a_p^2 = \frac{a^2F^2}{4R^2}\left(p_bc^2 + p_cb^2 - p_bp_ca^2 - p_cp_ab^2 - p_ap_bc^2\right) \tag{1}$$

Also, since quadrilateral AB_pPC_p cyclic with diameter R_a, by Sine Theorem we obtain:

$$a_p = R_a \sin A = R_a \cdot \frac{a}{2R} = \frac{aR_a}{2R}$$

By substitution $a_p = \frac{aR_a}{2R}$ in (1) we obtain barycentric representation for R_a^2:

$$R_a^2 = p_bc^2 + p_cb^2 - p_bp_ca^2 - p_cp_ab^2 - p_ap_bc^2 \tag{2}$$

Since $p_a = \frac{ad_a}{2F}$, $p_b = \frac{bd_b}{2F}$, $p_c = \frac{cd_c}{2F}$ then

$$p_b c^2 + p_c b^2 = \frac{bc^2 d_b}{2F} + \frac{b^2 c d_c}{2F} = \frac{bc}{2F}\left(c d_b + b c d_c\right)$$

and applying inequality $aR_a \geq c d_b + b d_c$ we obtain

$$p_b c^2 + p_c b^2 \leq \frac{abcR_a}{2F} = 2RR_a$$

Since $p_b = \frac{bd_b}{2F}$ and $p_c = \frac{cd_c}{2F}$ then

$$p_b p_c a^2 = \frac{a^2 b^2 c^2}{4F^2} \cdot \frac{d_b d_c}{bc} = 4R^2 \cdot \frac{d_b d_c}{bc}.$$

Thus

$$R_a^2 = p_b c^2 + p_c b^2 - p_b p_c a^2 - p_c p_a b^2 - p_a p_b c^2 \leq$$

$$\leq 2RR_a - 4R^2 \sum_{cyclic} \frac{d_b d_c}{bc} \Leftrightarrow (R - R_a)^2 \leq R^2 - 4R^2 \sum_{cyclic} \frac{d_b d_c}{bc} \Rightarrow$$

$$\Rightarrow \sum_{cyclic} \frac{d_b d_c}{bc} \leq \frac{1}{4}$$

and, therefore

$$a^2 yz + b^2 zx + c^2 xy = \sum_{cyclic} p_b p_c a^2 = \sum_{cyclic} \frac{bd_b}{2F} \cdot \frac{cd_c}{2F} a^2 =$$

$$= \sum_{cyclic} \frac{a^2 bc d_b d_c}{4F^2} = \frac{a^2 b^2 c^2}{4F^2} \sum_{cyclic} \frac{d_b d_c}{bc} =$$

$$= 4R^2 \sum_{cyclic} \frac{d_b d_c}{bc} \leq 4R^2 \cdot \frac{1}{4} = R^2$$

Second solution. We have (Stevin - Bottema) the following result:
If $x, y, z \in \mathbb{R}$ and a, b, c side lengths of triangle ΔABC and R circumradius of ΔABC, then

$$a^2 yz + b^2 zx + c^2 xy \leq R^2 (x + y + z)^2.$$

Taking account of $x + y + z = 1$, we obtain our result.

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Daniel Văcaru

W30. Solution by the proposer. Noting that

$$\ln\left(1+\frac{1}{2}\right)=\sum_{k=1}^{m}\frac{(-1)^{k-1}}{kn^k}+o\left(\frac{1}{n^m}\right)$$

we obtain that

$$n^m\ln\left(1+\frac{1}{n}\right)=n^m\sum_{k=1}^{m}\frac{(-1)^{k-1}}{kn^k}+n^m o\left(\frac{1}{n^m}\right)=$$

$$=\sum_{k=1}^{m}\frac{(-1)^{k-1}n^{m-k}}{k}+n^m o\left(\frac{1}{n^m}\right)=\sum_{k=0}^{m-1}\frac{(-1)^{k}n^{m-1-k}}{k+1}+$$

$$+n^m o\left(\frac{1}{n^m}\right)=P_{m-1}(n)+n^m o\left(\frac{1}{n^m}\right)$$

Since

$$n^m\ln\left(1+\frac{1}{n}\right)-P_{m-1}(n)=n^m o\left(\frac{1}{n^m}\right)$$

then

$$\min_{n\to\infty}\left(n^m\ln\left(1+\frac{1}{n}\right)-P_{m-1}(n)\right)=0$$

and, therefore,

$$\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n^m}}{e^{P_{m-1}(n)}}=e^{\lim\limits_{n\to\infty}\left(n^m\ln\left(1+\frac{1}{n}\right)-P_{m-1}(n)\right)}=1$$

Thus,

$$\left(1+\frac{1}{n}\right)^{n^m}\sim e^{P_{m-1}(n)}$$

W31. Solution by the proposer. First note that $x\neq 1$ and $y\neq 1$ because otherwise we get $x=y=1$. Let

$$D_< := \{(x,y)\,|x,y \in R_+, x<y \text{ and } x^y = y^x\}$$

Due to symmetry we have

$$\inf_{(x,y)\in D}(x-1)(y-1) = \inf_{(x,y)\in D_<}(x-1)(y-1)$$

Let $f(x) := x^{\frac{1}{x}}$, $x>0$. Since

$$f'(x) = \frac{f(x)}{x^2}(1-\ln x),$$

then $f(x)$ strictly increasing on $(0,e]$ and strictly decreasing on $(0,\infty]$ with $\max\limits_{x>0} f(x) = f(e) = e^{\frac{1}{e}}$.
Therefore, noting that

$$x^y = y^x \Leftrightarrow x^{\frac{1}{x}} = y^{\frac{1}{y}} \Leftrightarrow f(x) = f(y),$$

we can conclude that

$$(x,y)\in D_< \Rightarrow x<e<y$$

(otherwise if x,y both belong to $(0,e)$ or $(0,\infty)$ then, due to monotonicity of $f(x)$, equality $f(x)=f(y)$ yields $x=y$, that is the contradiction. And in case $x=e$, since $y\neq e$ we again obtain contradiction $f(e)=f(y)\neq f(e)$).
Also note that if $(x,y)\in D_<$ then $x>1$. Indeed, since

$$x^y = y^x \Leftrightarrow x = y\log_y x$$

and $y>e$ then supposition $x<1$ implies $0<\frac{x}{y}=\log_y x<0$, i.e. contradiction. Hence

$$(x,y)\in D_< \Rightarrow x\in(1,e),\, y\in(e,\infty)$$

Let $t := \log_x y - 1$. Then

$$\log_x y = t+1 \Leftrightarrow y = x^{t+1}$$

and

$$y = x\log_x y \Leftrightarrow y = x(t+1)$$

By substituting

$$y = x(t+1) \text{ in } y^x = x^y \Leftrightarrow \frac{y}{x} = x^{\frac{y}{x}-1}$$

we obtain

$$t+1 = x^t \Leftrightarrow x = (t+1)^{\frac{1}{t}} \Rightarrow y = (t+1)^{1+\frac{1}{t}}$$

where $t > 0$ (since $1 < x < y$). Thus,

$$D_< = \left\{ \left((t+1)^{\frac{1}{t}}, (t+1)^{1+\frac{1}{t}} \right) | t \in (0,\infty) \right\}$$

is set of all non-trivial solution of equation $x^y = y^x$, satisfied $x < y$
Let

$$H(t) := (x-1)(y-1) = (x-1)(x(t+1)-1) = x(x-1)(t+1) - x + 1$$

We have

$$H'(t) =$$

$$= (2x-1)x'(t+1) + x^2 - x - x' = x'((2x-1)(t+1)-1) + x^2 - x =$$

$$= x'(2x(t+1)-(t+2)) + x^2 - x =$$

$$= x(2x(t+1)-(t+2))(\ln x)' + x^2 - x =$$

$$= x\left((2x(t+1)-(t+2)) \left(\frac{\ln(1+t)}{t} \right)' + x - 1 \right) =$$

$$= x\left((2x(t+1)-(t+2)) \left(\frac{1}{t(1+t)} - \frac{\ln(1+t)}{t^2} \right) + x - 1 \right)$$

Since by Bernoulli inequality

$$(1+t)^{1+\frac{1}{t}} > 1 + \left(1+\frac{1}{t}\right)t = 2+t$$

then

$$2(1+t)^{1+\frac{1}{t}} - (t+2) > 4 + 2t - t - 2 > 0$$

and also, since $\ln(1+t) < t$, we have

$$\left(\frac{1}{t(1+t)} - \frac{a\ln(1+t)}{t^2}\right) + (t+1)^{\frac{1}{t}} - 1 > \left(\frac{1}{t(1+t)} - \frac{1}{t^2}\right) +$$

$$+ (t+1)^{\frac{1}{t}} - 1 = \left(\frac{1}{t(1+t)} - \frac{1}{t}\right) + (t+1)^{\frac{1}{t}} - 1 =$$

$$= \frac{1}{1+t} + (t+1)^{\frac{1}{t}} = (t+1)^{\frac{1}{t}} - \frac{t}{1+t} =$$

$$= \frac{(t+1)^{1+\frac{1}{t}} - t}{1+t} > \frac{1 + t\left(1+\frac{1}{t}\right) - t}{1+t} = \frac{2}{1+t} > 0$$

So, $H(t)$ increasing on $(0,\infty)$ and, therefore

$$(x-1)(y-1) = H(t) > \lim_{t\to 0} H(t) = (e-1)^2$$

Hence

$$\inf_{(x,y)\in D} (x-1)(y-1) = (e-1)^2$$

W32. Solution by the proposer. The series equals
$-2\zeta(2) + 3\zeta(3) - \frac{19}{4}\zeta(4)$.
We need Abel's summation by parts formula, which states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of real numbers and $A_n = \sum\limits_{k=1}^{n} a_k$, then
$\sum\limits_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum\limits_{k=1}^{n} A_k(b_k - b_{k+1})$, or

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} A_n b_{n+1} + \sum_{k=1}^{\infty} A_k(b_k - b_{k+1}). \qquad (1)$$

The following identity can be proved by mathematical induction.

$$\sum_{k=1}^{n} H_k = (n+1)H_{n+1} - (n+1), \quad n \geq 1.$$

We apply formula (1) with $a_n = H_n$ and
$b_n = \zeta^2(2) - \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right)^2 - \frac{2\zeta(2)}{n}$, and we have, based on formula (2), that