U332. Find $\inf_{(x,y)\in D}(x+1)(y+1)$, where $D=\{(x,y)|x,y\in\mathbb{R}^+, x\neq y, x^y=y^x\}$.

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Note that $e^{y\ln x}=x^y=y^x=e^{x\ln y}$ iff $\frac{\ln x}{x}=\frac{\ln y}{y}$, since e^x is a strictly increasing function for all real x. Note also that

$$\frac{d}{dx}\left(\frac{\ln(x)}{x}\right)=\frac{1-\ln(x)}{x^2}$$

is negative iff $x>e$, positive iff $x<e$, and zero iff $x=e$, ie for any $x\neq y$ such that $x^y=y^x$, we must have either $x>e>y$ or $x<e<y$. We may therefore define $D^*=D\cup\{(x,y)=(e,e)\}$, and the problem is equivalent to finding, for $(x+1)(y+1)$, either its minimum in D^* if it exists at $(x,y)=(e,e)$ (since it will coincide by continuity of functions $(x+1)(y+1),x^y,y^x$ with the infimum in D), or otherwise its infimum in D.

Define $f(x,y)=xy$, and $g(x,y)=x\ln(y)-y\ln(x)$. Note that the extrema of $f(x,y)$ subject to condition $g(x,y)=0$ may be found by Lagrange's multiplier method, or real constant λ exists such that

$$\lambda y=\ln(y)-\frac{y}{x},\qquad\qquad \lambda xy=x\ln(y)-y=y\ln(x)-x,$$

and since $x\ln(y)=y\ln(x)$, we find that a local extremum occurs iff $x=y$, ie the only local extremum of xy in D^* occurs when $x=y=e$, with a value e^2. Moreover, the borders of D^* occur when $x\to 1$ and $y\to\infty$ or *vice versa*, for $xy\to\infty$, or the extremum of xy at $x=y=e$ is a minimum with value e^2. We conclude that, for $(x,y)\in D^*$, we have

$$(x+1)(y+1)=xy+x+y+1\geq xy+2\sqrt{xy}+1\geq e^2+2e+1=(e+1)^2,$$

with equality iff $x=y=e$. By continuity of condition $x^y=y^x$ and of function $(x+1)(y+1)$, we conclude that

$$\inf_{(x,y)\in D}(x+1)(y+1)=(e+1)^2,$$

where $(x+1)(y+1)$ can get arbitrarily close to $(e+1)^2$ as $x>e$ gets arbitrarily close to e, with the corresponding value of $y<e$ getting arbitrarily close to e too, or *vice versa.*

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