U326. Find

$$\sum_{n=0}^{\infty} \frac{a_n + 2}{a_n^2 + a_n + 1}$$

where $a_0 > 1$ and $3a_{n+1} = a_n^3 + 2$ for all integers $n \geq 1$.

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Consider the series

$$\sum_{n=0}^{\infty} \frac{a_n + 2}{a_n^2 + a_n + 1}$$

where $3a_{n+1} = a_n^3 + 2$. The reduction of the series is as follows.

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{a_n + 2}{a_n^2 + a_n + 1} \\ &= \sum_{n=0}^{\infty} \frac{(a_n - 1)(a_n + 2)}{a_n^3 - 1} \end{aligned}$$

where $3a_{n+1} = a_n^3 + 2$ was used. Now,

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{a_n^2 + a_n - 2}{3(a_{n+1} - 1)} \\ &= \sum_{n=0}^{\infty} \frac{(a_n - 1)(a_n^2 + a_n - 2)}{3(a_n - 1)(a_{n+1} - 1)} \\ &= \sum_{n=0}^{\infty} \frac{a_{n+1} - a_n}{(a_n - 1)(a_{n+1} - 1)} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{a_n - 1} - \frac{1}{a_{n+1} - 1} \right). \end{aligned}$$

The collapse of this telescopic series leads to the desired result, namely,

$$\sum_{n=0}^{\infty} \frac{a_n + 2}{a_n^2 + a_n + 1} = \frac{1}{a_0 - 1}.$$

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