Undergraduate problems

U325. Let $A_1B_1C_1$ be a triangle with circumradius R_1. For each $n \geq 1$, the incircle of triangle $A_nB_nC_n$ is tangent to its sides at points A_{n+1}, B_{n+1}, C_{n+1}. The circumradius of triangle $A_{n+1}B_{n+1}C_{n+1}$, which is also the inradius of triangle $A_nB_nC_n$, is R_{n+1}. Find $\lim_{n\to\infty} \frac{R_{n+1}}{R_n}$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

U326. Find

$$\sum_{n=0}^{\infty} \frac{a_n + 2}{a_n^2 + a_n + 1},$$

where $a_0 > 1$ and $a_{n+1} = \frac{1}{3}\left(a_n^3 + 2\right)$ for all integers $n \geq 1$.

Proposed by Arkady Alt, San Jose, USA

U327. Let $(a_n)_{n\geq 0}$ be a sequence of real numbers with $a_0 = 1$ and

$$a_{n+1} = \frac{a_n}{n^2 a_n + {a_n}^2 + 1}.$$

Find the limit $\lim\limits_{n\to\infty} n^3 a_n$.

Proposed by Khakimboy Egamberganov, Tashkent, Uzbekistan

U328. Let $(a_n)_{n\geq 1}$ be an increasing sequence of positive real numbers such that $\lim\limits_{n\to\infty} a_n = \infty$ and the sequence $(a_{n+1} - a_n)_{n\geq 1}$ is monotonic. Evaluate

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n\sqrt{a_n}}.$$

Proposed by Mihai Piticari and Sorin Rădulescu, Romania

U329. Let $a_1 \leq \ldots \leq a_{\frac{n(n-1)}{2}}$ be the distances between n distinct points lying on the plane. Prove that there is a constant c such that for any n we can find indices i and j such that

$$\left|\frac{a_i}{a_j} - 1\right| < \frac{c \ln n}{n^2}.$$

Proposed by Nairi Sedrakyan, Yerevan, Armenia

U330. Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the properties

(i) f is an antiderivative,

(ii) f is integrable on any compact interval,

(iii) $f(x)^2 = \int_0^x f(t)\, dt$ for all $x \in \mathbb{R}$.

Proposed by Mihai Piticari, Câmpulung Moldovenesc, Romania