U261. Let $T_n(x)$ be the sequence of Chebyshev polynomials of the first kind, defined by $T_0(x) = 0$, $T_1(x) = x$, and

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

for $n \geq 1$. Prove that for all $x \geq 1$ and all positive integers n

$$x \leq \sqrt[n]{T_n(x)} \leq 1 + n(x-1).$$

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The formal definition of Chebyshev polynomials is as in the problem statement, but with $T_0(x) = 1$. We will actually use this definition in the following solution, since among other things, the definition with $T_0(x) = 0$ results after induction in $T_n(1) = n$ for all n, hence the second inequality would not hold. With the definition $T_0(x) = 1$, as we will soon see, both inequalities hold.

Assume that x is a fixed value, and for said x, consider the x-dependent sequence $(s_n)_{n\geq 0}$ defined by $s_0 = 1$, $s_1 = x$, and for all $n \geq 2$, $s_n = 2xs_{n-1} - s_{n-2}$. Using standard techniques, it follows that

$$s_n = \frac{\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n}{2} \geq \left(\frac{\left(x+\sqrt{x^2-1}\right) + \left(x-\sqrt{x^2-1}\right)}{2}\right)^n = x^n,$$

where we have used the power-mean inequality, which is valid since $x \pm \sqrt{x^2-1}$ are positive reals because $0 \leq \sqrt{x^2-1} < x$ for all $x \geq 1$. Note that equality holds iff equality in the power mean inequality holds, ie iff $\sqrt{x^2-1} = 0$, for $x = 1$, or iff $n = 1$ for all x.

The second inequality can be rewritten as

$$T_n(x) \leq (1 + n(x-1))^n .$$

For $n = 0$, both sides are identically 1, while for $n = 1$, both sides are identically x, or the inequality holds with equality for all $x \geq 1$ in these cases. For $n = 2$, the inequality rewrites as

$$2x^2 - 1 \leq (2x-1)^2, \qquad\qquad 2(x-1)^2 \geq 0,$$

clearly true, with equality iff $x = 1$. Now, we will show by induction that, for all $n \geq 2$ and all x, we have

$$T_n(x) = 2(x-1)\sum_{k=1}^{n-1}(n-k)T_k(x) + n(x-1)T_0(x) + 1.$$

For $n = 2$ and $n = 3$, this result is respectively equivalent to

$$T_2(x) = 2(x-1)T_1(x) + 2(x-1) + 1 = 2x^2 - 1,$$

$$T_3(x) = 2(x-1)T_2(x) + 4(x-1)T_1(x) + 3(x-1) + 1 = 2xT_2(x) - x,$$

clearly true in both cases. These are our base cases, and if the result holds for $n, n-1$, then

$$T_{n+1}(x) = 2(x-1)T_n(x) + 2T_n(x) - T_{n-1}(x) =$$

$$= 2(x-1)T_n(x) + 4(x-1)T_{n-1}(x) + 2(x-1)\sum_{k=1}^{n-2}(n-k+1)T_k(x) + (n+1)(x-1)T_0(x) + 1,$$

where we have used the hypothesis of induction for $n, n-1$, and the result clearly holds for $n+1$ too. Hence it holds for all positive integer n.

Now, this means that, if for some $n \geq 3$ the inequality holds for $1, 2, \ldots, n-1$, then

$$T_n(x) \leq 2(x-1) \sum_{k=1}^{n-1} (n-k) \left(1 + k(x-1)\right)^k + n(x-1) + 1,$$

with equality iff $x = 1$, since $T_2(x) = (1 + 2(x-1))^2$ iff $x = 1$, and if $x = 1$ then both sides are identically 1. Using the expression for the sum of the geometric progression with ratio $1 + n(x-1)$ from 1 to $(1 + n(x-1))^{n-1}$, we obtain

$$(1 + n(x-1))^n - 1 = n(x-1) \sum_{k=0}^{n-1} (1 + n(x-1))^k ,$$

or it suffices to show that

$$\sum_{k=1}^{n-1} (1 + n(x-1))^k \geq \sum_{k=1}^{n-1} \frac{2(n-k)}{n} (1 + k(x-1))^k .$$

If n is even, when $k = \frac{n}{2}$ the term in the LHS is $(1 + n(x-1))^{\frac{n}{2}}$, and in the RHS is $\left(1 + \frac{n}{2}(x-1)\right)^{\frac{n}{2}}$, clearly not larger, and equal iff $x = 1$. Whether n is odd or even, every k other than $\frac{n}{2}$ can be grouped in pairs of sum n, or it suffices to show that, for all integer k such that $1 \leq k < \frac{n}{2}$, we have

$$(1 + n(x-1))^k + (1 + n(x-1))^{n-k} \geq \frac{2(n-k)}{n} (1 + k(x-1))^k + \frac{2k}{n} (1 + (n-k)(x-1))^{n-k} .$$

Now, since $k < n - k < n$ for all such n, we have $1 + k(x-1) \leq 1 + (n-k)(x-1) \leq 1 + n(x-1)$, with equality iff $x = 1$, or it suffices to show that

$$\frac{n-2k}{n} (1 + n(x-1))^{n-k} \geq \frac{n-2k}{n} (1 + n(x-1))^k ,$$

clearly true since $n - 2k > 0$ and $1 + n(x-1) \geq 1$, with equality again iff $x = 1$.

The conclusion follows, equality holds in both equalities iff either $n = 1$ and for all $x \geq 1$, or $x = 1$ for all n.

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